VOL. 33, NO. 2, NOV.-DEC., 1959 PRICE 65¢

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

Address editorial, subscription and other business correspondence to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.

Manuscripts should be typed on 8½″× 11″ paper, double-spaced with 1″ margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simplest terms what the paper is about. Authors need to keep duplicate copies of their papers.

The MATHEMATICS MAGAZINE is published at Pacoima, California, by the managing editor, bi-monthly except July-August. Ordinary subscriptions are: 1 year $3.00; 2 years $5.75; 3 years $8.50; 4 years $11.00; 5 years $13.00. Sponsoring subscriptions are $10.00; single copies 65¢; reprints, bound 1¢ per page plus 10¢ each, provided your order is placed before your article goes to press.
Published with the assistance of the Mathematical Association of America.
Entered as second-class matter, March 23, 1948, at the Post Office, Pacoima, California, under act of Congress of March 8, 1876.

MATHEMATICS MAGAZINE

VOL. 33, NO. 2, NOV.-DEC., 1959

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IMPORTANT NOTICE

Henceforth please address all mail for the MATHEMATICS MAGAZINE to:

Robert E. Horton
Los Angeles City College
855 North Vermont Avenue
Los Angeles 29, California

As of December 1st., Mr. Horton is taking over the editing and publishing of the MATHEMATICS MAGAZINE. Because of his high personal qualifications, his experience in editing "Problems and Questions," and his enthusiasm for the aims of this magazine, I am delighted to have him take up this work which is so near to my heart.

Glenn James

Glenn James

AN ITERATIVE APPROXIMATION FOR FINDING THE N-TH ROOT OF A NUMBER

H. W. Gould

In the opening pages of G. H. Hardy's charming book *A Course of Pure Mathematics* it is left as an exercise to prove the following theorem.

If m and n are positive integers and m/n is a good approximation to $\sqrt{2}$, then the number $(m+2n)/(m+n)$ is a better approximation, and the errors in the two cases are in opposite directions.

For example, starting with $m = n = 1$ one finds the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \cdots$$

and the sixth fraction has the decimal value $1.4142857\cdots$ so that accuracy to four decimals is obtained.

Hardy suggests that the theorem may be generalized, but he does not unfold the history of this simple iteration process. The formula seems to have been known to Archimedes and to Hero of Alexandria, in a far more general setting.

It is the purpose of this paper to prove the above theorem by a mode of attack which shows how a mathematician is led to see a more general theorem. This brings to mind an important consideration for anyone teaching mathematics: It is not enough to write down a certain theorem or formula and prove it and go on to something else; one must somehow exhibit the motivation behind the formula or what brought it to mind in the first place. This "logic of discovery" is more important in some ways than the specific material covered.

In order to prove the theorem already stated, it is more convenient to work with a slightly more general statement as follows.

THEOREM 1. Let $1 < x < \sqrt{2}$ and $\sqrt{2} - x = \epsilon > 0$, and let $a = (x+2)/(x+1)$. Then the number a is a better approximation to $\sqrt{2}$ than is x in the sense that $a - \sqrt{2} < \epsilon$. Similarly, if it be assumed that $x - \sqrt{2} = \epsilon > 0$, then $\sqrt{2} - a < \epsilon$.

The original theorem is the special case when $x = m/n$, m and n being positive integers.

Proof. One has

$$a = \frac{x+2}{x+1} = 1 + \frac{1}{x+1},$$

so that

$$a - \sqrt{2} = \frac{1}{x+1} + 1 - \sqrt{2}$$

$$= \frac{1}{x+1} + \frac{(1-\sqrt{2})(1+\sqrt{2})}{1+\sqrt{2}}$$

$$= \frac{1}{x+1} - \frac{1}{1+\sqrt{2}}$$

$$= \frac{\sqrt{2}-x}{(x+1)(1+\sqrt{2})}$$

$$= \frac{\epsilon}{(x+1)(1+\sqrt{2})} < \frac{\epsilon}{4} < \epsilon,$$

which shows that the number a is indeed a better approximation, and one may even estimate roughly how much better an approximation it will be.

The second part of the theorem involves only a small reversal of inequalities and the proof follows the same pattern.

Now, the way in which the radicals worked out above suggests that one might possibly define

$$a-\sqrt{N} = \frac{1}{x+1} - \frac{1}{1+\sqrt{N}}$$

in order to discover a theorem for $\sqrt{N}$ in general.

This does not seem to work, but suggests that one might try letting

$$a-\sqrt{N} = C\left\{ \frac{1}{x+1} - \frac{1}{1+\sqrt{N}} \right\}$$

with the view of determining C in such a way that a neat formula may be had for the number a and such that a will be a better approximation to $\sqrt{N}$ than is x.

Let $\sqrt{N}-x = \epsilon > 0$.

Then one finds

$$a-\sqrt{N} = C\left\{ \frac{1}{x+1} - \frac{1}{1+\sqrt{N}} \right\}$$

$$= C\,\frac{\sqrt{N}-x}{(x+1)(1+\sqrt{N})}$$

$$= \frac{C}{(x+1)(1+\sqrt{N})}\,\epsilon .$$

Therefore, in order to have $a-\sqrt{N} < \epsilon$ one must require that

$$C \leqq d < (x+1)(1+\sqrt{N}) .$$

Let it be supposed that $1 < x < \sqrt{N}$, N being any positive real number > 1. Then x may be as small as 1 or as large as $\sqrt{N}-\epsilon$, and one may certainly then choose $C \leqq d = (1+1)(1+\sqrt{N}) = 2(1+\sqrt{N})$, so that the field of choice for C is wide. But one wishes to have a neat formula for the number a, and this leads one to explore the algebra of the situation a little more thoroughly. One finds

$$a-\sqrt{N} = \frac{C}{x+1} - \frac{C}{1+\sqrt{N}} = \frac{C}{x+1} - \frac{C(\sqrt{N}-1)}{(\sqrt{N}+1)(\sqrt{N}-1)}$$

$$= \frac{C}{x+1} - \frac{C(\sqrt{N}-1)}{N-1} = \frac{C}{x+1} - \frac{C\sqrt{N}}{N-1} + \frac{C}{N-1}$$

so that

$$a = \frac{C}{x+1} + \frac{C}{N-1} + \sqrt{N}\{1 - \frac{C}{N-1}\}.$$

This formula then suggests that a neat formula for a would be obtained by letting $C = N-1$ and then one would have

$$a = \frac{N-1}{x+1} + 1 = \frac{x+N}{x+1},$$

thereby giving a neat generalization of the formula in Theorem 1. To justify this choice of C, however, one must determine if this would lead to a better approximation. This is where the word "good" as applied to the term approximation has some importance. It may not be true that for all rough approximations x that a will be a better approximation with this choice of C. Now one must require that

$$C < (x+1)(1+\sqrt{N}),$$

so one has

$$N-1 < (x+1)(1+\sqrt{N}) = (\sqrt{N}+1-\epsilon)(1+\sqrt{N})$$

$$= (\sqrt{N}+1)^2 - \epsilon(1+\sqrt{N}),$$

$$= N + 2\sqrt{N} + 1 - \epsilon(1+\sqrt{N}),$$

or, therefore,

$$0 < 2\sqrt{N} + 2 - \epsilon(1+\sqrt{N},$$

so that

$$\epsilon(1+\sqrt{N}) < 2(1+\sqrt{N}),$$

which would surely be true if one required that $\epsilon < 2$. Thus, if one supposes that the approximation x is a *good approximation* to $\sqrt{N}$ in the sense that $\sqrt{N} - x = \epsilon < 2$, then the number defined by $a = (x+N)/(x+1)$ will be a better approximation to $\sqrt{N}$.

At this point the above reasoning may be summarized into compact form as follows.

THEOREM 2. Let $1 < x < \sqrt{N}$, where N is any positive real number > 1. Let $\sqrt{N} - x = \epsilon > 0$, and define $a = (x+N)/(x+1)$. Then the number a will be a better approximation to $\sqrt{N}$ whenever the approximation x is such that $\epsilon < 2$.

Having found a general result for square roots one naturally wonders next how to find a theorem for cube roots and n-th roots in general. The key so far seemed to lie in the process of rationalizing expressions containing radicals. Because of the elementary factorization formulas $x^2 - 1 = (x-1)(x+1)$ and $x^3 - 1 = (x-1)(x^2+x+1)$, it would seem plausible to look for a generalization, for example, in the form

$$a - \sqrt[3]{N} = C\{\frac{1}{1+x+x^2} - \frac{1}{1+\sqrt[3]{N}+(\sqrt[3]{N})^2}\},$$

for an appropriate choice of the number C, and with similar extensions for higher roots of N, by simply adding on more powers. The choice $C = N-1$ would lead to a neat formula for a.

THEOREM 3. In general, for sufficiently small values of ϵ, where $\epsilon = \sqrt[3]{N}-x$, the number $a = (x^2+x+N)/(x^2+x+1)$ will be a better approximation to $\sqrt[3]{N}$ than is the number x.

Proof. One has

$$a-\sqrt[3]{N} = C\left\{\frac{1}{1+x+x^2}-\frac{1}{1+\sqrt[3]{N}+(\sqrt[3]{N})^2}\right\}$$

so that

$$a = \sqrt[3]{N}+\frac{C}{1+x+x^2}-\frac{C(\sqrt[3]{N}-1)}{N-1}$$

$$= \frac{C}{1+x+x^2}+\frac{C}{N-1}+\sqrt[3]{N}\left\{1-\frac{C}{N-1}\right\},$$

so that by letting $C = N-1$ one would have the value of a given in the hypotheses of the theorem. Therefore one has

$$a-\sqrt[3]{N} = (N-1)\frac{1+\sqrt[3]{N}+(\sqrt[3]{N})^2-1-x-x^2}{(1+x+x^2)(1+\sqrt[3]{N}+(\sqrt[3]{N})^2)}$$

$$= (N-1)\frac{(\sqrt[3]{N}-x)+(\sqrt[3]{N})^2-x^2}{(1+x+x^2)(1+\sqrt[3]{N}+(\sqrt[3]{N})^2)}$$

$$= (\sqrt[3]{N}-1)\frac{(\sqrt[3]{N}-x)(1+\sqrt[3]{N}+x)}{1+x+x^2}$$

$$= (\sqrt[3]{N}-1)\frac{1+\sqrt[3]{N}+x}{1+x+x^2}\epsilon .$$

Therefore to obtain a better approximation it will be sufficient to require that

$$(\sqrt[3]{N}-1)(1+\sqrt[3]{N}+x) < 1+x+x^2 ,$$

or that is

$$(\sqrt[3]{N})^2 < 2+x(2-\sqrt[3]{N})+x^2 ;$$

$$(\sqrt[3]{N})^2+\left(\frac{2-\sqrt[3]{N}}{2}\right)^2-2 < x^2+(2-\sqrt[3]{N})x+\left(\frac{2-\sqrt[3]{N}}{2}\right)^2 ;$$

$$(\sqrt[3]{N})^2+1-\sqrt[3]{N}+\frac{(\sqrt[3]{N})^2}{4}-2 < \left(x-\frac{2-\sqrt[3]{N}}{2}\right)^2 ;$$

$$\frac{5}{4}(\sqrt[3]{N})^2-\sqrt[3]{N}-1 < \left(\frac{3}{2}\sqrt[3]{N}-1-\epsilon\right)^2 .$$

From this last inequality one could require therefore that

$$\epsilon < \frac{3}{2}\sqrt[3]{N} - 1 - \sqrt{\frac{5}{4}(\sqrt[3]{N})^2 - \sqrt[3]{N} - 1}$$

in order to assure a better approximation. For example if $N = 8$ this relation requires that $\epsilon < 3$. Certainly any first approximation one chooses will generally be better than that, and so in general one may expect that the formula in this theorem applies for cube roots.

Now to generalize to n-th roots of N one would suspect that taking

$$a = \frac{x^{n-1} + x^{n-2} + \cdots + x^2 + x + N}{x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1}$$

would give a better approximation than x. The algebraic arguments here could be strengthened so as to obtain such a result. However it is interesting at this point to look at the geometry of the situation.

It is the ancient Rule of Double False Position (regula duorum falsorem) which affords a neat generalization of the formulas discussed so far.

Consider an interval of real numbers, say $x_1 \leqq x \leqq x_2$ and suppose a function f is defined and continuous on this interval. If $f(x_1)$ and $f(x_2)$ have opposite algebraic signs, then the function f must possess at least one zero, $x = x_0$, in this interval. A simple graph is shown in Figure 1 to

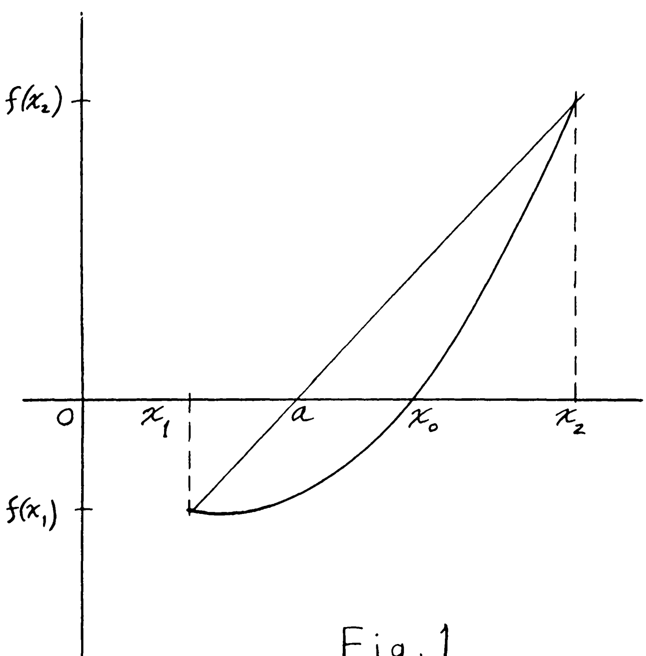

Fig. 1

suggest the geometry here. If the function f does not differ too far from a linear function, or is what one might call "reasonably well-behaved", then one might expect that a straight line drawn from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ would cross the horizontal axis closer to the root x_0 than x_1 or x_2. Let the value of x for which the straight line crosses the axis be the number a. Then from the slopes of the segments involved one has at once that

$$\frac{a-x_1}{-f(x_1)} = \frac{x_2-a}{f(x_2)},$$

so that

$$a = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2)-f(x_1)}.$$

One may calculate this number and if it is not zero, then it may be seen that one could repeat the process using a and either x_1 or x_2 depending on the signs and hope to eventually converge closer on some root of the function. If one assumes that the function has only one root in the interval considered then the method will always converge on that root. In Figure 2 are shown some typical convergents to a root.

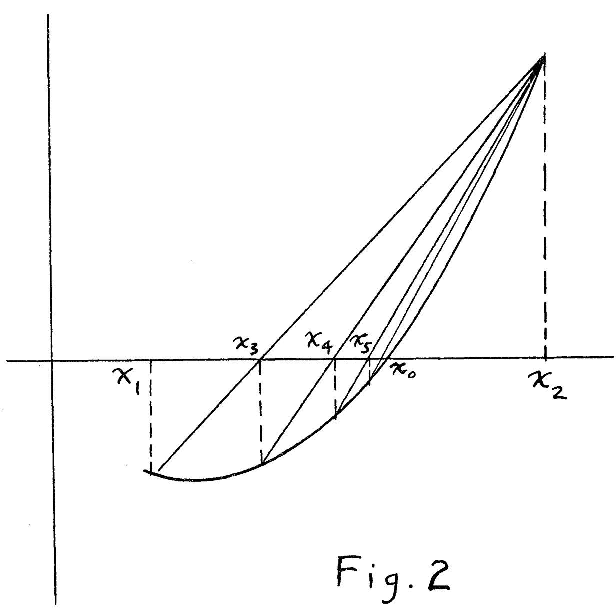

Fig. 2

The formula for the number a just written may be rewritten as follows:

$$a = x_2 - \frac{f(x_2)}{\frac{f(x_1)-f(x_2)}{x_1-x_2}},$$

and from this, by taking a limit as x_1 approaches x_2, one has the famous Newton-Gregory approximation :

$$a = x_2 - \frac{f(x_2)}{f'(x_2)},$$

which sometimes yields a better approximation. "Sometimes" – because one must assume differentiability, and also that $f'(x_2)$ is relatively large. The Newton-Gregory method is a tangent-line approximation rather than a chordal or secant-line approximation and so the curve may bend and fluctuate in such a manner as to give worse approximations than the first rough approximation. This is notoriously so if the derivative is nearly zero. In Figure 3 an example of this situation is shown where the second and third approximations are farther and farther away from the desired root.

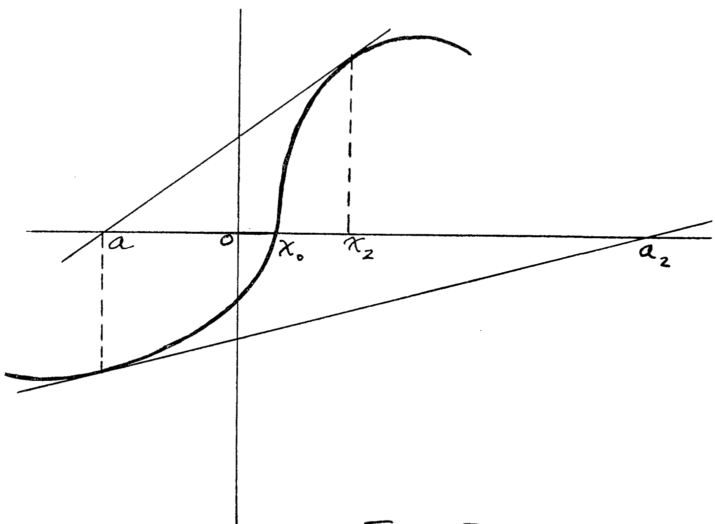

Fig. 3

Now in the formula for a for Double False Position, let f be defined by $f(x) = N - x^2$ and take $x_1 = 1$. One then obtains a picture something like that shown in Figure 4. The formula for the number a becomes also

$$a = \frac{(N-x_2^2)\cdot 1 - x_2(N-1^2)}{(N-x_2^2)-(N-1^2)} = \frac{N-x_2^2-x_2N+x_2}{1-x_2^2} = \frac{N+x_2}{1+x_2}$$

which is exactly the relation given in Theorem 2.

For large values of N, of course, it would be better to choose some other value than 1 for x_1, since that would then be a very poor approximation and so from this point of view Theorem 2 leaves much to be desired. On the other hand the formula there for the second approximation is easy to remember and apply.

In the case of n-th roots one proceeds in a similar fashion. Let $f(x) = N - x^n$, and suppose $a < b$ and $f(a)$ and $f(b)$ to be of opposite algebraic

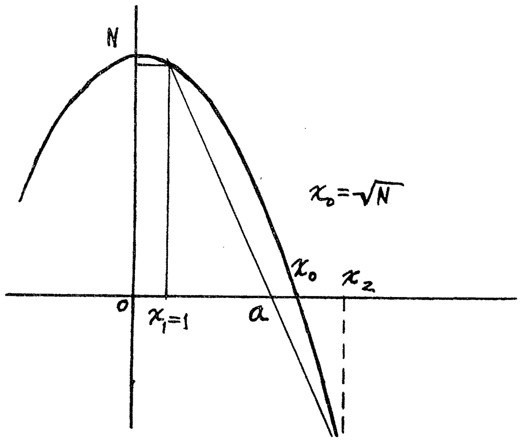

Fig. 4

sign. Then one has for a second approximation, say c, that

$$c = b - \frac{N-b^n}{\dfrac{N-a^n-N+b^n}{a-b}} = b + \frac{N-b^n}{\dfrac{a^n-b^n}{a-b}},$$

and then because of the algebraic identity

$$\frac{a^n-b^n}{a-b} = \sum_{k=0}^{n-1} a^k b^{n-k-1},$$

the formula for c reduces ultimately to

$$c = \frac{N+b^{n-1}a+b^{n-2}a^2+\cdots+b^2a^{n-2}+ba^{n-1}}{b^{n-1}+b^{n-2}a+b^{n-3}a^2+\cdots+ba^{n-2}+a^{n-1}}.$$

This expression then generalizes all the previous formulas. In particular the formula suggested from the algebraic considerations is obtained when $b = 1$. The form of such an expression as has been discussed here is easily conjectured on the basis of *algebraic* manipulations, however the *geometry* of the situation suggests in this case more general formulas whose validity is more easily checked.

In the above expression for c one would not choose $b = 1$ in order to have a rapidly converging iteration. One would choose a and b each very close to the desired root and then the number c would always lie between the two and be a better approximation.

In the formula just given for c, let $n = 3$. Then a formula is obtained which is sometimes called Hero's formula. For a very interesting discussion of this formula and what it has to do with Diophantine equations (such as Pell's equation) reference might be had to Tobias Dantzig's recent book *Bequest of the Greeks*. Iterative approximations turn up rather naturally in the study of continued fractions and Diophantine equations.

The binomial theorem has been a fertile source of iterative formulas for n-th roots. For some examples of this, an article by Otto Dunkel (American Mathematical Monthly, Vol. 24, 1927, pp. 366-368) may be consulted.

All of these iterative approximations have their individual attractions, some because they are easy to remember, others of a more complicated nature because of their rapidity of convergence. In each instance it is most instructive to keep in mind the interplay between algebra and geometry. As old as the rule of double false position is, it is still the most widely applicable method for approximating to roots, and so perhaps it is the central landmark in this particular chapter of mathematics.

West Virginia University
Morgantown, West Va.

$f(x) = x^2 - 9.$

3 is a zero of $f(x)$.

Sir! Three is not zero! Is it?!

Well! ...

HARMONIC INVERSION

Sahib Ram Mandan

ABSTRACT

The purpose of this paper is to introduce the idea of Harmonic Inversion w. r. t. a pair of complementary sub-spaces in an n-dimensional space and then deduce a number of properties as an immediate consequence of the definition of this operation, e. g. a pair of mutually self-polar simplexes invert harmonically into another such pair. The existence of Moebius Simplexes, that are mutually interlocked, i. e., inscribed as well as circumscribed to each other, is established in a space of an odd number of dimensions. Successive inversions w. r. t. the vertices of a simplex and its respective opposite prime faces form a cycle, whereas all the inversions w. r. t. all the pairs of opposite elements of a simplex together with identity form a group leading to a set of 2^n associated points such that any quadric variety, for which the simplex is self-polar, passing through any one of these points, passes through all of them. Finally it is shown that all the reflections, w. r. t. all the axial elements of a rectangular system of axes, together with the inversion w. r. t. the origin of the system and identity form a group.

1. *INTRODUCTION.*

In a space n of n dimensions, let x & x' be two complementary sub-spaces of dimensions x & x' respectively in the sense that they have no point common with each other, and $x+1$ independent points of the sub-space x and $x'+1$ of x' together constitute $n+1$ independent points that determine the given space n, i. e., $n = x+x'+1$. Let P be a general point in the space n but outside the spaces x & x', let the unique secant from P to them meet them in P_x & $P_{x'}$ respectively, let P' be a point on this secant such that the cross-ratio $(PP_xP'P_{x'}) = \rho$: Thus a (1, 1) correspondence is set between the pairs of points P, P', if ρ is fixed, i. e., to every point P corresponds a point P' and to every point P' corresponds a point P'' other than P, provided $\rho = -1$ in which case P and P' correspond to each other, i. e., the correspondence is reversible or involutory, for they are now separated harmonically by the spaces x & x'. In the latter case P and P' are said to be *harmonic inverse* of each other, P is said to *invert harmonically*, or we shall simply say, P *inverts*, into P' and vice versa w. r. t. the spaces x & x'. Thus the idea of Harmonic Inversion is introduced in a general space w. r. t. a pair of its complementary sub-spaces.

2. *PROPERTIES.*

The following properties follow immediately, when harmonic inversion is performed in the space n w. r. t. the spaces x & x':

a. *The reference spaces x & x' are self-inverse,* for every point of either space inverts into itself w. r. t. them.

b. *A line inverts into a line.* There arise two cases here according as the given line meets or not either of the spaces x & x', unless it is a secant to them in which case too, evidently, it inverts into itself. In the first case, the inverse line meets the space x or x' where the given line does, and the two lines determine a plane that has common, a line through their common point, with one of the spaces, and a point only with the other. In the second case, the inverse line is a generator of the solid quadric generated by the secants from the points of the given line to the spaces x & x', which have each a generator common with this quadric.

c. As an immediate consequence of the preceding property, we have another, viz., *Cross-ratio of a range or a pencil is unaltered, and therefore their harmonic relations are too preserved.*

d. *Any surface of any order inverts into another of the same order, and therefore any curve of any order inverts into another of the same order.* In particular, *any space of any dimensions inverts into another of the same dimensions,* e. g., *a plane or a hyper-plane inverts into another plane and hyper-plane respectively, a quadric inverts another one.*

e. *A pair of conjugate points for a quadric invert into a pair of conjugate points for the inverse quadric,* therefore, *a pair of reciprocal simplexes for a quadric invert into a pair of reciprocal simplexes for the inverse quadric, and a self-polar simplex for a quadric inverts into a self-polar simplex for the inverse quadric,* for, the pair of conjugate points for a quadric are separated harmonically by the pair of points where their join meets the quadric.

f. *Tangency is preserved,* and therefore, *a pair of conjugate hyper-planes for a quadric invert into a pair of conjugate hyper-planes for the inverse quadric,* in fact, *any pair of polar or conjugate sub-spaces for a quadric invert into a pair of polar or conjugate sub-spaces for the inverse quadric,* for, the pair of conjugate hyper-planes for a quadric are separated harmonically by the pair of tangent hyper-planes to the quadric through their meet, i. e., through their common $(n-2)$-space.

g. *The quadrics for which the apaces x & x' are polar to each other invert into themsleves,* for the secants to x & x' meet them in pairs of points separated harmonically by the pairs of points where the quadrics cut these secants.

h. *The polar line*[1] *of a point P in a plane w. r. t. a triangle therein inverts into the polar line of the point P', inverse of P, w. r. t. the* *inverse triangle;* and therefore, *the polar plane*[1] *of a point P in a solid space w. r. t. a tetrahedron therein inverts into the polar plane of the point P′, inverse of P, w. r. t. the inverse tetrahedron;* and hence, *the polar prime*[2] *of a point P in a four dimensional space w. r. t. a simplex therein inverts into the polar prime of the point P′, inverse of P, w. r. t. the inverse simplex.* Extending successively the definition of a polar prime of a point P in a space of n dimensions w. r. t. a simplex therein as one through the n polar $(n-2)$-spaces of the projections of P from the vertices of the simplex into its respective opposite prime faces w. r. t. the corresponding simplexes therein, we may state that : *The polar prime of a point P in a space of n dimensions w. r. t. a simplex therein inverts into the polar prime of the point P′, inverse of P, w..r. t. the inverse simplex.*

k. As an immediate consequence of the preceding property, we have: *Any pair of mutually self-polar triangles*[1], *tetrahedra*[1] *or simplexes*[2] *invert into another such pair.**

3. *MÖBIUS SIMPLEXES.*

Let $S(A) \equiv A_0 A_1 \cdots A_{n-1} A_n$, $S(B) \equiv B_0 B_1 \cdots B_{n-1} B_n$ be a pair of reciprocal simplexes w. r. t. a quadric Q_{n-1} in the space n and the spaces x, x' (Art. 1) be a pair of its skew generators. Let harmonic inversion be performed w. r. t. x & x', Q_{n-1} inverts into itslef, for the secant from any point of it lies wholly on it; the simplexes $S(A)$ & $S(B)$ therefore invert into a pair of its reciprocal simplexes $S(A')$ & $S(B')$, A_i' $(i = 0, 1, \cdots, n-1, n)$ being inverse of A_i and B_i' that of B_i. Then A_i is conjugate to A_i' and therefore lies in its polar prime $B_0' B_1' \cdots B_{i-1}' B_{i+1}' \cdots B_{n-1}' B_n'$ w. r. t. Q_{n-1}. Similarly lies B_i' in the prime $A_0 A_1 \cdots A_{i-1} A_{i+1} \cdots A_{n-1} A_n$. Thus the simplexes $S(A)$ & $S(B')$ are mutually interlocked, i. e. one is circumscribed as well as inscribed to the other. Similarly too is then the relation between $S(A')$ & $S(B)$. Such pair of simplexes may be termed after Moebius following such a term for a pair of Moebius Tetrahedra[3]. But the existence of Moebius Simplexes depends upon that of a pair of skew generators x & x' of Q_{n-1} which is possible[4] only when n is odd, say equal to $2m+1$, in which case the spaces x & x' are both of dimensions m. Hence:

There exist Moebius Simplexes, that are mutually interlocked, i. e., circumscribed as well as inscribed to each other, in space of an odd number of dimensions only.

4. *CYCLE OF INVERSIONS.*

a. Let U_0 be the unit point $(1, 1, \cdots, 1, 1)$ referred to a simplex $S(A)$

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* The existence of mutually self-polar simplexes in spaces of dimensions higher than four is yet to be established.

(Art. 3). Let the harmonic inverse (h. i.)* of U_0 w. r. t. the vertex A_0 of $S(A)$ and its opposite prime face be U_1 whose co-ordinates referred to $S(A)$ will be $(-1, 1, \cdots, 1, 1)$. Let U_2 be the h. i. of U_1 w. r. t. the vertex A_1 of $S(A)$ and its opposite prime face $\cdots$. Let U_{i+1} be the h. i. of U_i w. r. t. the vertex A_i of $S(A)$ and its opposite prime face in succession so that the final point U_{n+1}, after $n+1$ successive inversions w. r. t. the vertices of $S(A)$ and its respective opposite prime faces, will be $(-1, -1, \cdots, -1, -1)$, referred to $S(A)$, that coincides with the starting point U_0. Thus : *The $n+1$ successive harmonic inversions, w. r. t. the vertices of simplex and its respective opposite prime faces, as operations, form a cycle. The result may be observed geometrically too.*

b. Extending the idea of Umbilical projection[5] and that of inversion and orthogonality in regard to hyper-spheres in spaces of dimensions higher than four we may deduce now the following proposition :

The successive $n+1$ inversions w. r. t. the $n+1$ mutually orthogonal hyper-spheres in a space of n dimensions form a cycle.

5. *GROUPS.*

a. *Each harmonic inversion is a self-inverse operation,* i. e. one repetition of this operation w. r. t. the same element brings us back to the starting point. If this operation is denoted by "0" and its repetition by its product to itself, i. e. $0 \cdot 0$ or by its square, i. e., 0^2, then $0^2 = I$ where I denotes identity in the language of groups.

b. Let us denote the harmonic inversions w. r. t. the vertices A_i $(i = 0, 1, 2)$ and the respective opposite sides of triangle $A_0A_1A_2$ by "i" respectively and operate upon the unit point $U_0 \equiv (1, 1, 1)$ referred to this triangle. Then U_0 inverts into U_1 $(-1, 1, 1)$, U_2 $(1, -1, 1)$ and U_3 $(1, 1, -1)$ by the operations 0, 1, 2, respectively. Now if we invert any one of these four points by any one of these three operations, we get at no new point but one of them only other than the one considered showing that the resultant of any two of these operations in any order is equivalent to the remaining third and that of all the three in any order is equivalent to the identity. Thus : *The three harmonic inversions w. r. t. the vertices and the respective opposite sides of a triangle together with the identity form a commutative or an Abelian Group of order four* with the following table:

	0	1	2
0	I	2	1
1	2	I	0
2	1	0	I

* (h. i.) stands for "harmonic inverse" in what follows.

The three self inverse operations, "i" constitute a set of generators and $i^2 = I = 0 \cdot 1 \cdot 2$ that of generating relations[6].

c. Let $i = 0, 1, 2, 3$ denote the harmonic inversions w. r. t. the vertices A_i and the respective opposite faces of a tetrahedron $A_0A_1A_2A_3$ respectively. Let U_1 $(-1, 1, 1, 1)$, U_2 $(1, -1, 1, 1)$, U_3 $(1, 1, -1, 1)$, U_4 $(1, 1, 1, -1)$, U_5 $(-1, -1, 1, 1)$, U_6 $(-1, 1, -1, 1)$, U_7 $(-1, 1, 1, -1)$ be the h. i. of the unit point $U_0 \equiv (1, 1, 1, 1)$, referred to this tetrahedron, w. r. t. the operations 0, 1, 2, 3, 4, 5, 6, respectively, where the three new operations 4, 5, 6 are the resultants of the pairs of operations (0, 1) or (2, 3), (0, 2) or (3, 1), (0, 3) or (1, 2), in any order, respectively equivalent to the harmonic inversions[3] w. r. t. the three corresponding pairs of opposite edges[1] of the tetrahedron, viz., (A_0A_1, A_2A_3), (A_0A_2, A_3A_1), (A_0A_3, A_1A_2). Now any one of the above eight points inverts into one of them only other than the one considered by any one of the above enumerated seven operations. Thus : *The seven harmonic inversions w. r. t. the seven pairs of opposite elements of a tetrahedron together with the identity, form an Abelian Group of order eight, with the following table :*

	0	1	2	3	4	5	6
0	I	4	5	6	1	2	3
1	4	I	6	5	0	3	2
2	5	6	I	4	3	0	1
3	6	5	4	I	2	1	0
4	1	0	3	2	I	6	5
5	2	3	0	1	6	I	4
6	3	2	1	0	5	4	I

The four self-inverse operations "i" constitute a set of generators and $i^2 = I = 0 \cdot 1 \cdot 2 \cdot 3$ that of generating relations. Here we may further notice from the above table that the three inversions 4, 5, 6 together with identity, form a subgroup of Index 2 and therefore self-conjugate[6] too. Thus : *The three harmonic inversions w. r. t. the three pairs of opposite edges of a tetrahedron together with identity form a self-conjugate subgroup of Index two.*

d. Following the line of argument of the preceding paragraph we may now simply state that : *The fifteen harmonic inversions w. r. t. the fifteen pairs of opposite elements of a Simplex in a four dimensional space together with identity form an Abelian Group of order sixteen with the* following table :

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	I	5	6	7	8	1	2	3	4	14	13	12	11	10	9
1	5	I	9	10	11	0	14	13	12	2	3	4	8	7	6
2	6	9	I	12	13	14	0	11	10	1	8	7	3	4	5
3	7	10	12	I	14	13	11	0	9	8	1	6	2	5	4
4	8	11	13	14	I	12	10	9	0	7	6	1	5	2	3
5	1	0	14	13	12	I	9	10	11	6	7	8	4	3	2
6	2	14	0	11	10	9	I	12	13	5	4	3	7	8	1
7	3	13	11	0	9	10	12	I	14	4	5	2	6	1	8
8	4	12	10	9	0	11	13	14	I	3	2	5	1	6	7
9	14	2	1	8	7	6	5	4	3	I	12	13	10	11	0
10	13	3	8	1	6	7	4	5	2	12	I	14	9	0	11
11	12	4	7	6	1	8	3	2	5	13	14	I	0	9	10
12	11	8	3	2	5	4	7	6	1	10	9	0	I	14	13
13	10	7	4	5	2	3	8	1	6	11	0	9	14	I	12
14	9	6	5	4	3	2	1	8	7	0	11	10	13	12	I

The five self-inverse operations, $i = 0, 1, 2, 3, 4$, constitute a set of generators and $i^2 = 0\cdot 1\cdot 2\cdot 3\cdot 4 = I$ that of generating relations.

The thirty one harmonic inversions w. r. t. the thirty one pairs of opposite elements of a simplex in a five-dimensional space together with identity form an Abelian group of order thirty two. The fifteen harmonic inversions w. r. t. the fifteen edges and the respective opposite solid spaces of the Simplex together with identity, form a self-conjugate subgroup of index two.

The sixty three harmonic inversions w. r. t. the sixty three pairs of opposite elements of a Simplex in a six-dimensional space together with identity form an Abelian Group of order sixty four.

The 127 harmonic inversions w. r. t. the 127 pairs of opposite elements of a Simplex in a seven-dimensional space together with identity, form an Abelian of order 128.

The 63 harmonic inversions, w. r. t. the 28 edges and the respective opposite five dimensional spaces, and w. r. t. the 35 pairs of opposite solid spaces, together with identity form a self-conjugate subgroup of index two.

e. We are now in a position to make a general statement as follows : *The* $2^n - 1$ *harmonic inversions w. r. t. the* $2^n - 1$ *pairs of opposite elements of a Simplex in an n-dimensional space together with identity form* *an Abelian Group of order* 2^n.

The $2^{n-1}-1$ *harmonic inversions w. r. t. the* $2^{n-1}-1$ *pairs of opposite elements both of odd number of dimensions of the simplex together with identity form a self-conjugate subgroup of index two and this is possible only when n is odd.*

6. *A SET OF* 2^n *ASSOCIATED POINTS.*

Any quadric variety, in an n-dimensional space, for which a given simplex is self-polar, inverts harmonically into itself w. r. t. all the pairs of opposite elements of the simplex (Art. 2g). If such a quadric passes through a given point U_0, it will then pass through all the 2^n-1 h. i. points of U_0 w. r. t. the 2^n-1 pairs of opposite elements of the simplex. Thus : *A given point and all its h. i. points w. r. t. all the pairs of opposite elements of a simplex in an n-dimensional space, constitute a set of* 2^n *associated points such that any quadric variety, for which the simplex is self-polar, passing through any one of these points, passes through all of them.*

For a 5-space reference be made to the paper, "Baker's property of the weddle surface" by W. L. Edge published in the Journal of London Mathematical Society, Vol. 32, 1957, pp. 463-66.

7. *REFLECTIONS.*

Let a prime face of a simplex in an n-dimensional space, recede to infinity such that it is self-polar for the hyper-sphere at infinity. Then the simplex converts into a rectangular system of axes with origin at the vertex opposite the prime face at infinity, and the harmonic inversions w. r. t. its pairs of opposite elements into reflections, w. r. t. the corresponding elements of the system, that include the inversion w. r. t. the origin. Thus : *The* 2^n-2 *reflections w. r. t. the* 2^n-2 *axial elements of a rectangular system, in an n-dimensional space, together with the inversion w. r. t. the origin of the system and identity form an Abelian group of order* 2^n. *The* $2^{n-1}-1$ *reflections w. r. t.* $2^{n-1}-1$ *axial elements, of odd number of dimensions, of the system, together with identity form a self-conjugate subgroup of index two and this is possible only when n is odd (Art. 5e).*

The 2^n *associated points (Art. 6) here form the vertices of a rectangular parallelotope or an orthotope.*[6]

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Indian Institute of Technology
Khargpur, India

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York.*

LEARNING THEORIES AND THE MATHEMATICS CURRICULUM

Robert E. Horton

Mathematicians have given much thought recently to possible revisions and improvements in the mathematics curricula in schools and colleges. Any revision of the curriculum should be made in harmony with the best that is known about how students learn. Consequently it seems desirable to make available in one place those aspects of psychological research in the area of learning theory which bear most directly upon the learning of mathematics.

There are several major current theories on how learning takes place. As no one of them is universally accepted, it seems desirable to take a somewhat eclectic point of view for the purposes of curriculum revision. Any attempt to justify a curriculum item on a psychological basis would be done in reference to that learning theory which seemed to be the most reasonable support, yet was not in complete disagreement with other theories. T. R. McConnell writing in the *Forty-First Yearbook of the National Society for the Study of Education* pointed out that there were large areas of agreement that can be synthesized from the differing theories (14). Thus the use of such areas of agreement provides the best support available in the absence of one universally accepted learning theory. Drawing from the *Twenty-First Yearbook of the National Council of Mathematics* (15) and the *Forty-First Yearbook of the National Society for the Study of Education* (16), a brief summary of the pertinent aspects of the major theories of learning appears.

Conditioning. – Although this theory seems best fitted to explain how rote memorization and fact learning comes about, it has implications about practice and repetition that bear upon concept learning. Learning, according to the conditioning theory, is accomplished by association. The principle of association is stated by Guthrie as: "... *a stimulus pattern that is acting at the time of a response, will, if it recurs, tend to produce that* *response*" (10:23). Howard F. Fehr (9) points out that conditioning relates to the making and breaking of habits and the acquisition of skills. Conditioning is less capable of explaining more abstract thinking. The response to a pattern of stimuli is conditioned. Learning proceeds from activity. Incorrect responses are learned as well as correct responses. Conflicting and inhibitory stimuli give rise to new responses. Learning occurs normally in one conditional response. A skill is not a simple thing but is a large collection of habits. Therefore, a need arises for repetition in learning skills. Learning is most effective when a desired response is associated with appropriate signs, gestures, mathematical symbols, and words that act as stimuli for the desired action. As doing is required for learning we must be free to act. Learning takes place best in a situation where the learner is free to act rather than in a restricted situation.

Connectionism.—As a theory of learning, connectionism provides for a better understanding of the processes of trial and error that occur in problem solving situations. Peter Sandiford (19) lists several features of connectionism that are important in studying concept formation. He writes:

> Connectionism boldly states that learning is connecting. The connections presumably have their physical basis in the nervous system, where the connections between neuron and neuron explain learning.
>
> ...
>
> Some connections are more natural than others. ... we mature into reflexes and instincts, but we have to practice or exercise in order to learn our habits.
>
> ...
>
> According to connectionism those things we call intellect and intelligence are quantitative rather than qualitative affairs. A person's intellect is the sum total of the bonds he has formed. (19:98)

One important aspect of connectionism as a theory of learning is its belief that learning is greatly influenced by inherited characteristics of the individual. Learning occurs when a bond is established between a stimulus and the response made by the organism. By trial and error methods these bonds develop a pattern according to Thorndike's laws of effect, exercise, and readiness (19:112).

Fehr summarizes the principal characteristics of the connectionist learning theory as:

> 1. Thinking back to similar situations to find a particular response that worked previously; the transfer of identical elements.
> 2. Trial and error, discarding unsuccessful paths (responses); avoiding wrong responses.
> 3. Each complex situation is to be broken up into a series of simple elements arranged in a sequential order. Each simple element is mastered separately. The seriated set of mastered elements make up the whole.
> 4. After the whole situation is obtained, repeat and drill until the solution is sufficiently strengthened (conditioned) for later recall.
> 5. Reward successful learning of desired goals. (9:17)

Because of its atomistic approach to learning, connectionism seems to lack the power to explain and provide for the functions of abstracting and generalizing that are so important in the more abstract processes of concept formation.

Field theories. – The field theorist believes that organisms experience situations as wholes rather than receiving discrete stimuli. Therefore, the holistic field theory of learning is in sharp contrast to the atomistic conditioning and connectionism theories. According to the field theories, learning occurs when we observe the whole of any situation as we respond. We respond in a generalized, interacting field rather than to an external stimulus. Learning progresses in terms of relatedness between facts and the entire situation. Field theories also explain learning in terms of insight. That is, the falling into place of the various elements of a situation to provide a meaningful whole. Insight is the successful, understanding behavior of an organism in a situation. Analysis of the part-whole relationship leads to a thought process of the trial and error type, but better expressed by the term approximation and correction leading to the goal. Seeing relationships between parts and the whole leads to generalization and thinking that can be transferred to new situations.

Fehr (9:18) points out that field psychology explains learning more adequately than earlier theories. Initial learnings derive from experience rather than from definitions. The dynamic aspects of events are aids to learning. Whatever is to be learned must emerge from a problem situation that challenges or motivates the individual. To learn, we must see the problem as a whole. Isolation of details prevents insight. The analysis of part-whole relationships and viewing these in terms of past experience may lead to the reconstructing of the entire situation. This occurrence is called insight. Abstraction and generalization are predominant. This analysis and insight gives meaning to whatever is being learned, especially in mathematics. Either a lack of experience or overemphasis on the narrow habit formation may prevent insight.

After insight has been achieved, repeated practice will clarify the learning. Forgetting the new learning is less likely to occur when the knowledge is formalized and systematized. The whole learning situation is always part of a larger whole. Relationships that appear in one context appear again in greater generalization in larger configurations. The structure of knowledge is organized by processes of analysis, synthesis, and deductive logic. For learning to proceed we must have systematized knowledge rather than isolated facts.

T. R. McConnell found the following areas of agreement among the contrasting learning theories:

> 1. Both situation and response are complex and patterned phenomena.
>
> 2. Descriptions and interpretations of learning as of all aspects of behavior, must be made in terms of the mutual relationships among events rather than in terms of independent properties of the parts.

3. The organism must be motivated to learn.

4. Responses during the learning process are modified by their consequences.

5. Motivation is the direction and regulation of behavior towards a goal.

6. So-called trial-and-error behavior might be more appropriately described as a process of 'approximation and correction' or of 'trying this-and-that lead to the goal.'

7. Learning is essentially complete ... when the individual has clearly perceived the essential relationships in the situation and has mastered the fundamental principle involved in the concrete problem.

8. The transfer of learning from one situation to another is roughly proportional to the degree to which the situations are similar in structure or meaning.

9. Discrimination, as well as generalization, is an essential aspect of effective learning. (14:243-279)

Learning as reorganization. – One further theory is presented here because of its direct bearing on mathematics learning. The reorganization theory is considerably more limited in psychological scope than are the preceding theories. What it attempts to show is that learning may proceed most effectively if attention is focused at the outset on limited goals rather than on the ultimate goal of mature understanding. Thus it may be well to permit some responses that would inhibit achieving the final goal, then proceed to remove the inhibiting factors.

William A. Brownell (4) presents experimental evidence that a "crutch" whose continued use would inhibit maximum learning actually aids in early learning. Furthermore, the use of the crutch may later be discarded without difficulty. He explains these findings in terms of a learning theory of reorganization. He says:

> What one does when one learns is to attack the new problem with whatever reactions are available. These reactions are seldom if ever of the blind trial-and-error variety, but represent forms of behavior which have been connected previously with some aspect of the problem situation. Thus at the very outset of learning we encounter organization of behavior, and not random or purely chance responses. The degree of success which attends the application of the first behavior organization determines ... whether it will be retained and practiced, or rejected in place of another group of reactions ... . As soon as some reasonably adequate reaction pattern is found, it is accepted as a solution. (4:78)

Brownell explains the reorganization of learning phase in the following terms:

> So long as the reaction pattern satisfied all the needs of the situation as this is envisaged by the learner, the pattern will continue to be used, with ever increasing efficiency. But new needs may present themselves in the situation ... . Continued (repetitive) use of the old pattern will not, because it cannot, yield the newly needed behavior organization. What is required is a reorganization of behavior at a higher or more mature or more expert level. When this is achieved and first put to work, performance may be clumsy, slow and faulty. Practice, however, removes these bars to efficiency,

and behavior at the new level in time becomes rapid, easy and accurate. (4:79)

Brownell points out that this theory implies that the process of learning is one of organization and reorganizing on successively higher levels of our behavior. The basic aspect of learning is not practice but creation of a series of reaction patterns, each of which in turn gives way to a new pattern at a higher level of organization. The reorganization takes place when for any reason the given reaction pattern is unsatisfactory. For example a child may learn to count on his fingers. Finger counting will be discarded when it proves to be too slow for the handling of larger numbers. Or he may discard it when it is seen to be juvenile and his peers are using more sophisticated methods. On a higher level the students are taught to differentiate functions by the delta process. However, use of this process is discarded when it proves to be inefficient in handling complicated functions.

I. CONCEPT FORMATION

Acquisition of certain mathematical concepts is one of the primary purposes for which a student pursues a given curriculum. The word "concept" occupies a central position in much of the discussion about curriculum matters. Therefore, a clarification of the meaning of this word is necessary. First, concepts are not sensory data, but they result from sensory experiences which are mentally collected, generalized and developed. Second, concepts tend to elicit the same response as the sensory data from whence they are derived. That is, a person tends to react to the word "Fire!" in the same way that he would react to the actual fire. Third, the concept forms through a process in which the sense impressions become symbolized. That is, words or perhaps mathematical symbols are the mental products of the process by which the sense impressions are integrated into a concept. Fourth, there seems to be a hierarchy of concepts. Some concepts appear to be only slightly removed from sensory experience and their origin in sensory experience is readily traced. However, other concepts are so abstract that any tracing of their sensory origin is difficult. There seem to be certain very abstract concepts that may not depend upon sensory experience at all. Fifth, the attainment of a concept by a person seems to provide him with a selective device by which external stimuli are classified in order to create particular symbolic responses. Sixth, although most concepts are symbolic, some concepts may exist in a sub-verbal level.

The proceding discussion of the word "concept" follows the line of reasoning presented by Henry Van Engen in an article entitled "The Formation of Concepts". (22) In addition, some thought should be given to mathematical concepts. First, mathematical concepts vary widely in the hierarchy of concepts from relatively concrete to very abstract. For example, the concept of addition is relatively concrete and seems to stem from sense impressions of collections of physical objects. On the other hand, the concept of mathematical induction is very abstract and may well be a mental construct not derived directly from sensory experience. Second, a concept represented by a symbol may change and expand greatly in significance for the individual as his experience progresses. For example, the concept of equal may relate only to positive integers in the mind of a child. Yet, as he experiences more training in mathematics, the same symbol may take on a wide variety of connotations related to many aspects of both concrete and abstract equivalence relationships. Third, mathematical concepts exhibit a hierarchy of inclusiveness. That is, some concepts include and depend upon the awareness of subordinate concepts. For example, the concept of a number continuum depends upon understanding the concepts of rational and irrational numbers as well as other concepts.

Summarizing this discussion can best be done by quoting W. Edgar Vinacke:

> They [concepts] must be regarded as selective mechanisms in the mental organization of the individual, tying together sensory impressions, thus aiding in the identification and classification of objects. But concepts involve more than the integration of sense impressions, against the background of which recognition occurs, for they are linked with symbolic responses which may be activated without the physical presence of external objects. That is, concepts can be given names – can be detached from specific instances, by means of words – and used to manipulate experience over and beyond the more simple recognition function. The symbolic response, however, stands for whatever it has been linked within the previous experience of the organism and depends upon how that past experience is organized. (23:5)

To clarify further the meaning of the word, some additional features will be mentioned. Concepts are not direct sensory data but are the results of such data after they have been combined, analyzed, elaborated, classified, et cetera. Concepts are mental constructs. The concepts that an individual has are a result of the previous experience of the individual and the mental organization of these experiences achieved by the individual. Concepts are responses which organize, or link or combine discrete sensory experiences. It is assumed that such organization or combination is achieved through symbolism. The same symbolic response may result from a variety of data. In the internal mental processes of the individual, concepts represent selective factors. That is, discrimination is accomplished by means of concepts. Behavior related to concepts may be of two types. Either an external stimulus may arouse a symbolic response or a symbolic response may direct perceptual responses. For example, a diagram on the blackboard may evoke the mental response "triangle", or the mental response "root" may set an individual to searching for all the numbers which will make a polynomial vanish.

One further aspect of concepts is important for our mathematical considerations. Vinacke states:

> ... concepts have both horizontal and vertical organization and that, in consequence, the same object (or relation) has different points of reference depending upon the other objects with which it is compared. Horizontally, objects may be classified into

different categories, all of them equally inclusive. At the same time, objects may be classified vertically, into groups of varying complexity, or into more and more inclusive categories. (23:4)

Theories of concept formation.—One of the definitive studies of concept formation was conducted by Kenneth L. Smoke. He defines concept formation as follows:

> By "concept formation," "generalization," or "concept learning" we refer to the process whereby an organism develops a symbolic response (usually, but not necessarily, linguistic) which is made to the members of a class of stimulus patterns but not to other stimuli.
>
> Such a generalized symbolic response is not a "common element." (20:8)

Vinacke (23:6) points out that there are two types of concept formation. In one case an individual may be the passive recipient of a set of sensory impressions which his mind gradually organizes into a concept. On the other hand, a person may build a concept by establishing an hypothesis and then proceeding to check it against instances. These two processes are probably interrelated but they may be considered separately for discussion purposes. He calls the theories concerning these two types of concept formation the Composite Photograph Theory and the Active Search Theory.

Woodworth describes the composite photograph theory in these terms:

> The features common to a class of objects summate their impressions on the observer, who thus gradually acquires a picture in which the common features stand out strongly while the variable characteristics are washed out. (25:801)

This theory is related to the process of abstraction. It is possible that the concept may not be the common elements but rather the relationship between the stimulus patterns.

The Active Search Theory is described by Woodworth as, "The concept is supposed to originate as a hypotheses, which *O* proceeds to test by trying it on fresh specimens of the class" (25:801). This theory emphasizes the process of generalization. It presumes that the individual plays a more active, participating role in the conscious development of the concept.

In his experimental study of concept formation, Smoke lists the following findings:

> 1. Individuals who have learned concepts may be unable to give an accurate verbal formulation of them.
>
> 2. The process of concept formation seems to involve grouping.
>
> 3. Insightful behavior seems to be present in at least some instances of concept formation.
>
> 4. Concept formation, like most "thinking" appears to involve the formulation, testing and acceptance or rejection of hypotheses.
>
> 5. It seems probable that ability to learn concepts rapidly is correlated positively with ability to make high intelligence-test scores. (20:44)

The acquisition of concepts.—In his review of the research pertaining to concept formation, Vinacke (23:16) found several conditions related to the acquisition of concepts that are important for the purposes of the present study of mathematical concept formation. He found that concept formation was related to the individual's age or level of maturity. Even the process of concept formation seemed to be different in children and in adults. The types of concepts acquired varied with age. In particular, time concepts were achieved relatively late.

In connection with intelligence, Vinacke states:

> It is probably safe to say that psychologists have assumed that intelligence and concept formation are related, although they have not yet worked out the relationship explicitly. There are two assumptions, really: (1) that one of the variables of intelligence is the ability to form and use concepts, and (2) that part, at least, of the reason why mental age increases during the period of growth is that the ability to conceptualize increases. (23:18)

Training and experience only complicate the study of the relationship between concept formation and intelligence. The ability to form concepts seems to correlate more highly with experience, as measured by school grade level, than with intelligence quotients.

Vinacke found that the ability to form concepts had very little relation to the socio-economic status of the individual. However, a factor which did have substantial positive correlation with concept formation was the vocabulary of the individual.

Teaching to produce concepts.—We are concerned with effective ways of teaching to produce mathematical concepts in freshman engineering students. Research on efficiency in producing concepts is therefore of importance to this study. Vinacke (23:20) found that the order in which concepts were presented was a factor affecting the speed and efficiency of learning concepts. The order of simple-to-complex concepts was more effective than presenting concepts in the order of complex-to-simple. He also found that giving the concept to the student outright is no more effective than requiring the student to extract the concept from a series of concrete instances.

Concepts seem to be acquired more easily if the features or elements which identify the concept are accessible perceptually and are more "thing like." When the individual is allowed to participate directly in the learning situation, concept formation occurs more rapidly than when he is a passive observer in the situation. It appears that when more misleading or negative characters are introduced into a concept learning situation, the learning is less effective. The presence of negative instances may even inhibit the learning process.

When the name of a concept, its meaning, and information about the nature of the concept are introduced, learning is more effective than when the name only is presented. Keeping the goal of concept learning in the mind of the learner makes him a more active participant in the problem situation of grouping, testing hypotheses, and selecting or rejecting them. The proportion of consistent concepts achieved seems to increase as the series of learning instances is lengthened. Finally, as concepts become more complex, there is a tendency for thinking to become less logical, incorrect concepts to increase, and correct concepts to decrease.

Student behavior in concept formation.—In the concept learning situations, Vinacke reported several kinds of typical behavior exhibited by the learner. First, concepts are acquired in a gradual manner. He states:

> ... In general the subject is exposed to material and little by little begins to interpret it—perhaps by perceiving in it familiar objects or associating aspects of it with his past experience; perhaps by deliberately attempting to analyze it, comparing one instance in the experimental situation with another, etc. In time some features begin to stand out, others to "disintegrate", until the subject is able to formulate a principle which will work. (23:23)

An order of attainment of concepts seems to be a characteristic of student behavior noted by Vinacke. He found that students first attained concepts of objects, then of forms, then of number. In most of the experiments reported the concept of number appeared last in the order of attainment. Furthermore, it appears that it is difficult to alter the order in which concepts are formulated by the individual.

Vinacke reports three levels of student performance in concept formation. The first is a primitive "concrete" performance in which the grouping of elements is more or less random. The second is an intermediate performance which shows some but not all of the aspects of conceptual thinking. Finally at the advanced conceptual performance level the student exhibits the comprehension of the task of classification of stimuli into a concept.

II. MEMORY VERSUS HIGHER MENTAL PROCESSES

Traditional educational practice placed great emphasis on memory and the memorizing of facts and procedures. Yet many modern educators have raised questions about the effectiveness of memorizing in producing the kind of learning that we want to take place in our schools and colleges. Justification for the use of memorization was formerly made on the basis that memorizing produced effective learning. John Dewey and others questioned whether memorizing facts actually produces success in developing the higher mental processes.

The higher mental processes.—To commence this discussion, a definition of terms is necessary. By memory is meant the mental capacity or process of retaining and reviving impressions, or of recalling or recognizing previous experiences. The term "higher mental processes" is used in the manner described by Ralph W. Tyler (21:7). Higher mental processes involve: (1) the recall of principles taught and their application to situations that had not been presented during the course, and (2) the drawing of inferences from data that had not been presented before. It is clear that under these definitions, the problem of concept formation belongs in the realm of higher mental processes rather than in the realm of pure memorization.

In recent years, experimental evidence has become available to shed some light on the question of whether memorizing facts produces success in exercising the higher mental processes. At Ohio State University, Tyler found:

> ... in none of the widely varying courses in which tests were given was there a perfect relation between recall of information taught in the course and recall and application to new situations of principles learned in the course. ..., the relatively low correlations show clearly that application is a mental process different from mere recall. (21: 11)

Tyler's findings also revealed a somewhat lower correlation between drawing inferences from new data and the mere recall of facts. He found that of the students placing in the highest ten per cent of their classes in recall of facts taught in the course, forty per cent were below average in drawing inferences from new data. Tyler concludes from his findings:

> ... Memorization of facts frequently fails to result in the development of higher mental processes. If the higher mental processes of application of principles and inference are really to be cultivated, learning conditions appropriate for their cultivation are necessary. (21: 17)

The distinctions between meaningful learning and memorization were studied extensively by George Katona (13). He found that although memorizing is a quicker method of learning when only a small amount of material is involved, for large amounts and for more complex material meaningful learning is not only easier but more acceptable to the learner. Also learning by memorizing requires a considerable amount of over-learning if the material is to be available for recall after a long interval. Over-learning is the repetition of the memorized stimulus beyond the point of first mastery. Thus memorizing may prove slower than meaningful learning if retention over a long period is desired. Memorizing was found to permit transfer of training from one material to another to only a very limited extent. Conversely, those who pursued meaningful learning often found no more difficulty in applying principles to new material than to deal with the practiced material.

In describing meaningful learning, Katona states:

> The subjects proceed to discover or to construct the solution, and the preceding training helped them in doing so. Reproduction was not at all similar to a door bursting open, because a button had been pressed – it did not consist of the presentation of an ever-ready response to the appropriate stimulus. It was more like the process of discovery, of problem solving, and of construction. Remembering can here be best characterized as a rediscovery – a *reconstruction*. The effect of learning was ability to reconstruct. (13:43)

Developing the higher mental processes. – Concept formation is seen to be one of the higher mental processes when we examine Katona's description of the way in which the higher mental processes are developed. He states three results of his investigation.

> 1. Learning by memorizing is a different process from learning

> by underatanding.
>
> 2. Learning by understanding involves substantially the same process as does problem solving — the discovery of a principle.
>
> 3. Both problem solving and meaningful learning consist primarily in changing, or organizing, the material. (13:53)

The organization of the material is the activity carried on when the mind establishes or finds and understands a principle. Thus learning by understanding involves grouping material in such a manner that the intrinsic relationships are apprehended. As meaningful learning is primarily organization of material, it follows that remembering of meaningful material is reconstruction. This kind of remembering is the re-doing of the process of organization.

Katona believes that it is not the specific materials that are retained in meaningful learning. Rather, those specific materials are only examples used by the mind in extracting the principle involved. The memory of the subject who pursues meaningful learning does not consist of the specific examples. It does consist of the essential points and principles involved in the task. "We do not learn examples; we learn by examples." With the aid of examples the learner gains a knowledge of relations or qualities of the material and the essential point of the problem. Katona describes these as "whole-quantities" or qualities of the gestalt. Meaningful learning is a development of whole-quantities.

Transfer of training proceeds best using the meaningful learning method. Katona states:

> With meaningful learning most highly developed, we find that there is a plasticity or flexibility of learning which permits reasonable applications on a wide scale. These applications are due, not to the transfer of specific data from an *A* to a *B*, but to the fact that meaningful learning results in the acquisition of integrated knowledge, which is usable under different circumstances. (13:136)

When meaningful learning results in the development of whole-qualities then is applied to a new situation, the elements of the situation changed but the whole-qualities, principles, or essence of the situation are recalled by the memory. The new problem is attacked and solved in terms of the whole-qualities. When our learning is designed to preserve whole-qualities (concepts) rather than elements, then these may be applied successfully under different circumstances.

Retention of well-understood tasks. — One of the main aspects of learning is the reduction of forgetting. Katona's research showed that under meaningful learning the usual Ebbinghaus curve of forgetting did not prevail. In fact, "... retention of well-understood tasks may remain high even without intermediate practice" (13:144). Meaningful learning involves insight and may come at some stage of the process "like a flash." However, meaningful learning is often a gradual step-by-step process in which repetition of the elements occurs many times. Each repetition contributes to the clarification of the elements leading to a whole-quality. Consecutive examples strengthen and widen the subject's understanding of the principles being learned.

The retention of meaningful learning is described by Katona as follows:

> ... A student of mathematics who has understood the main points of a new hypothesis, may have the hypothesis at his command several months later, even though he may not have thought of it in the mean time. ... Additional practice of a different kind may strengthen an integrated knowledge by enriching it and by widening its scope. The reverse of this statement holds true: understanding is that form of learning which can be perfected by different kinds of application. (13: 163)

It appears that after meaningful learning we remember organized whole-qualities rather than specific elements. When we wish to apply our memory to the solution of a new problem, we select specific items from our whole-quality that will bear upon the problem.

Katona makes it clear that not all kinds of experience creates meaningful learning. Memorization establishes learning which is limited in its range of applications. An experience in which organization and understanding occurred leads to meaningful learning which may be applied to achieve organization and understanding in a new situation. Katona finally concludes that memorizing is not the prototype of learning. He believes that further research will establish that understanding organized wholes is the prototype of learning. That is, that true learning occurs only when whole-qualities of a situation are perceived and organized into a meaningful structure of thought.

III. RELATIONAL, SYMBOLIC, AND ABSTRACT THINKING

In the preceding section, the term "higher mental process" was discussed in contrast to memorization. It is desirable to examine in more detail some aspects of these higher mental processes as they are characteristic of the type of thinking required in the more advanced mathematics studied by engineers. It is clear that mathematical analysis requires pupil behavior of a more complex type than mere reaction to sensory impressions. The mind, as conceived by Judd, must enter into the behavior in an active and directive manner. In this connection, we might give another definition of the term "higher mental process." It is a mental process in which the mind of the individual makes a large contribution through its own conscious effort. This is in contrast to a low level mental process in which the mind reacts to sense impressions perhaps with little or no consciousness of its own action. Here the major contributor to behavior is external. In the higher mental processes, the mind is the major contributing factor in the behavioral pattern. For the purpose of this discussion, the processes of comparing, symbolizing, and abstracting will be examined.

Relational thinking. – The memory of an individual may record a series of sense impressions which may be recalled when desired. However, the individual may wish to contrast or compare some of these impressions with others. A conscious process of comparison may be carried out as the mind functions under its own initiative. Judd points out, "The individual who makes a comparison contributes much which the outer world does not supply" (12:19). In contrast, memorized material is determined primarily by what is given in the external situation. Judd continues:

> ... Memory supplies what may be thought of as the raw materials, but the mind, in setting up a connection between one item and another, as it must in application and inference, draws on past and present experience for more than a single contribution. Where memory supplies several items, it presents these in a series which is hardly more than a temporal sequence. The higher mental processes associate, or integrate, the items and thus build up a new subjective whole. (12:19)

Although Judd seems to be writing here in terms of a mind-matter dualism, one can still talk in terms of levels of mental activity while considering the mind as a function operating in a field.

Judd showed that expository and discursive writing contained a high percentage of relational words. He concluded that this emphasis on relational expressions implies that as thinking moves into higher levels, "there is a marked trend away from purely substantive contemplation toward combination and recombination of substantive ideas" (12:26). The higher mental processes involve the synthesis of ideas. The mind combines isolated sense impressions into integrated systems which can be utilized in the processes of application and inferring. It is in this kind of mental activity that relational thinking is important.

Symbolic thinking.—Relational thinking depends upon the ability of the mind to express relations in terms of verbal symbols. The higher mental processes involve both the synthesis activity characteristic of relational thinking and the activity of analysis. One is analyzing when he selects from among a large variety of sense impressions some particular element which may characterize the situation. Both synthesis and analysis are the results of activity of the mind. As Judd puts it:

> ... The impressions which the individual receives through his senses are sifted, selected, and organized into new combinations. The final outcome of all these processes is intelligent understanding of the world in terms of categories of organization which are contributed by the individual. (12:41)

Symbolic comprehension must be distinguished from symbolic verbalism. In mathematics, many a student has memorized symbols, tables of addition and multiplication, and other symbolic data. Yet he may be unable to apply these symbols for useful purposes or draw logical inferences from them. Such students have developed symbolic verbalism without symbolic comprehension. For example, freshman calculus students often develop a sort of conditioned response, "integrate" whenever the term "area" is applied as a stimulus. However, these same students may have little understanding of the true relation of the definite integral to what is defined as the area under a given curve. The fact is that it requires repetition and reorganization of experience by the individual before he can comprehend the real meaning of many of the symbols that he uses.

Judd asserts that by means of symbolic thinking the individual is able to transcend experience. That is, by understanding the laws of some symbolic system he is able to think beyond his personal concrete experience. The necessity for organization and reorganization of experience to achieve this symbolic power is stated by Judd as follows:

> At the higher levels of arithmetical thought and manipulation as well as at the lower levels, it is not enough that the mind acquire mere rules or successions of isolated ideas. There must be an organization of experience of a form that is to be described by the term "conceptual." No one ever fully understands the concept number until the perceptual world has been left far behind and organized abstract experiences have been built up which are totally different in character and purpose from concrete experiences. (12:72)

Abstract thinking.—Abstract thinking requires the ability to generalize. One must focus attention on some unique phase of experience which can be discovered through analysis to exist in a variety of situations. This is more than just locating any common element, although that process is involved in abstract thinking.

Ernest R. Breslich emphasizes that mathematics beyond the level of elementary arithmetical calculation involves abstract thinking. He states :

> Similarly, pupils have to cultivate a high degree of competency in abstract thinking to comprehend the fact that algebraic formulas deal with general functional relations and that geometrical theorems apply to space relations without reference to the different substances which take on spatial forms. (3:77)

In concluding the discussion of relational, symbolic and abstract thinking a few generalizations may be made.

1. Elementary mental processes are composed primarily of reactions to external sensory stimuli. Higher mental processes involve internal mental activity. The higher the level of mental process the less it is conditioned by external sense impressions.

2. Symbolic thinking is important because as a substitute for experience it allows one to transcend concrete experience. By means of symbols, the reorganization of experience is facilitated. Symbolic ideas can be used without keeping in mind the many concrete sense impressions which gave rise to the symbolic concept.

3. The higher mental processes are carried on by means of integrated systems of experience rather than by accumulations of discrete items of experience. When individual experiences are organized and integrated into a higher system, this system takes on characteristics not appearing in the original experiences. That is, an integrated whole concept is more than the sum of its parts.

4. Because the integration of experience is subjective, the nature of their form in different individuals will vary widely. This variation results from innate differences in persons and from their widely differing background of experience.

IV. PROBLEM SOLVING

In any course in mathematics, one of the goals is to develop in the student the ability to solve problems. The psychology of problem solving is therefore of importance in the development of mathematics curricula.

Definition of the term "problem."—One of the most broadly inclusive definitions of the word "problem" is given by John Dewey. He says:

> ... if we are willing to extend the meaning of the word *problem* to whatever—no matter how slight and commonplace in character—perplexes and challenges the mind so that it makes belief at all uncertain, there is a genuine problem or question involved in this experience of sudden change. (7:13)

Dewey emphasized that problems grow out of experience and are integral parts of experience. A problem appears when some blocking of behavior occurs for which the individual knows no direct means of removing.

Further clarification of the word "problem" appears in a statement by Henderson and Pingry:

> ... the necessary conditions for the existence of a problem-for-a-particular-individual:
> 1. The individual has a clearly defined goal of which he is consciously aware and whose attainment he desires.
> 2. Blocking of the path toward the goal occurs, and the individual's fixed patterns of behavior or habitual responses are not sufficient for removing the block.
> 3. Deliberation takes place. The individual becomes aware of the problem, defines it more or less clearly, identifies various possible hypotheses (solutions), and tests these for feasibility. (11:230)

Brownell (5:416) points out that true problems occupy positions somewhere in the middle of a continuum ranging from puzzles at one end and completely familiar and understandable situations at the other. Problems are not "puzzles." These he defines as bewildering situations whose successful solution occurs by accident and which do not result in learning which can be transferred to other puzzles.

Definition of problem solving.—Brownell says, "One may define problem solving so broadly as to make the term synonymous with learning" (5:415). He goes on to say, "There is little disposition nowadays to attribute problem-solving behavior to the activity of some special faculty or department of mind" (5:421). Thus we can think of problem solving as a normal activity of the mind which is almost indistinguishable from the general process of learning. Brownell feels that one important aspect of the individual's behavior in problem solving is its purposive character. It is not random or haphazard action but is directed toward the removal of a block to some goal. In the argument over trial-and-error versus insight as the rational process in solving problems, Brownell tends to believe that insight is the correct explanation. However, he warns against attributing some mystical quality to insight. He grounds his concept of insight in the reorganization of previous learning.

The process of problem solving.—Dewey (7:107-116) described reflective thinking in terms that apply to the solving of problems. He isolated for the purposes of discussion five mental activities: recognition of a blocking, intellectualization of the difficulty, identification of various hypotheses, testing these hypotheses and finally selection of one of the hypotheses and acting upon it. Henderson and Pingry (11:237) recognize only three aspects of problem solving: orientation to the problem, producing relevant thought material, and testing hypotheses.

In any case, problem solving can be related closely with what is termed the scientific method. Peterson states:

> ... scientific inquiry is defined as a method which utilizes the data gathered by the sense apparatus from experience and then subjects these to empirical justification. The process is one of induction whereby one investigates particular instances for the purpose of drawing general but tentative rules for application. (17:208)

Research in problem solving processes of college students shows that successful problem solving students differ from those who are unsuccessful in four major classifications. Bloom and Broder list these as differences in:

1. Understanding of the nature of the problem.
2. Understanding of the ideas contained in the problem.
3. General approach to the solution of problems.
4. Attitude toward the solution of problems. (1:25)

Utilizing the research of Katona (13), Brownell (4) (5), Bloom and Broder (1), Boyd (2) and Polya (18), some general conclusions about problem solving can be drawn. Problem solving that involves understanding is superior to problem solving with a memorized procedure. Problem solving is an example of one of the higher mental processes. Whether problem solving will be of the understanding type or the pure memory type depends in large measure on the kind of instruction given. Instruction which aids the pupil to organize and reorganize his previous experience produces a higher grade of problem solving techniques.

V. A PSYCHOLOGICAL BASIS FOR CURRICULUM DEVELOPMENT

It seems possible to organize the implications of psychology for curriculum development into three broad areas: scope, sequence, and transfer of learning.

Scope. — By scope here is meant the amount that should be in the curriculum and how it should be organized. First we can conclude that the curriculum should be organized so that the learner is a participant rather than a spectator. Buswell states, "All theorists agree in emphasizing the importance of response in learning. In the last analysis a curriculum is what it causes pupils to *do*" (6:448). Here we must include in the term *doing* such activities as the exercise of the higher mental processes as well as observable muscular behavior.

Second, the curriculum should relate to the experience background of the pupil and at the same time be organized in a systematic way. On this point, Buswell says:

> Psychologically there are merits in both the life-situation and the systematic-organization positions. Furthermore they are supplementary, rather than mutually exclusive, concepts. It is only when either position wants a monopoly on the curriculum that trouble arises. The solution to the problem must be derived from sensible, working relationship between the two concepts. (6:450)

Third, the curriculum should provide for the organization and reconstruction of experience. Beginning with Dewey (8) and carried on by Brownell (4) and others, the trend in learning theories appears to be toward the organization of experience as the basis for learning rather than toward the basing of learning upon the accumulation of experiences. Furthermore, the organization of experience should be carried to the point of the development of abstractions, generalizations, principles and laws.

Fourth, the curriculum must be organized in both a longitudinal and lateral fashion. That is, a student needs to have an appropriate series of experiences introducing, motivating, generalizing, and applying concepts. But he also needs to have these experiences related to other parallel developments. He should be able to place them in their proper context in the larger aspects of his environment. Buswell feels that the two-factor plan of curriculum development, life-situation, and systematic-organization plans would preserve the logical structure of content as well as provide excellent motivation through life situations.

Sequence.—The mathematics curriculum probably provides less flexibility in sequential organization than most other areas of learning. Yet there are implications from psychological research regarding even the mathematics sequence. We have already noted the statement by Vinacke (23:20) that concept learning proceeds best in the order of simple-to-complex. This order is not necessarily the same order as the logic of mathematics would take. For example, we start the teaching of arithmetic somewhere in the middle of its logical structure. Then we move in the direction of more complex arithmetical operations. Later we move in the opposite direction toward the basis of our number system to study the foundations of arithmetic. This is certainly in contrast to our usual sequence of study in plane geometry where we start with the basic elements and always move in the direction of greater complexity. In other words, in geometry the teaching sequence follows the axiomatic development of the mathematics, while in arithmetic it does not. Yet both sequences conform to the psychological principle of simple-to-complex *concepts*.

In discussing this lack of flexibility in mathematical order, Wheat states:

> To learn arithmetic, the pupil follows a set pattern of thinking. We, his teachers, do not invent the pattern. The situations of his everyday life do not dictate it. The pupil does not choose it according to his tastes or adapt it to his individual peculiarities and predilections. He adapts himself to the pattern. It is for us merely to recognize what the pattern requires and to see that the pupil recognizes and respects the requirements. (24:24)

What Wheat has said about arithmetic applies with somewhat less rigidity to higher branches of mathematics. Yet it is true that each concept in calculus, for example, has certain prerequisite concepts. This definitely limits the amount of alteration in sequence that is feasible. As Buswell says:

> The main point is that good learning depends on a coherent sequential organization of a curriculum and that a mere summation of facts or experiences in random order will not satisfy. (6:459)

In experimental situations, Katona (26) has investigated the role played by order in which material is learned. If two items A and B are learned, then the temporal order is either AB or BA. Katona studied the differences in learning outcomes for the orders AB and BA when A and B represent different kinds of learning experiences.

In situations where both learning processes A and B consisted of memorization, no significant differences were found between the outcomes of AB and BA in the strength of the facilitating or inhibiting factors. For most practical purposes, it seems that the two learning sequences were equivalent.

Katona found, however, that when one of the learning processes was characterized by the understanding of well-organized material (A) and the other by the acquisition of enumerated information (B), then the learning sequence AB facilitated the learning of B to a greater extent than the sequence BA. Katona stated:

> The sequence in which understanding comes first was found to be superior to the reverse sequence, both when reproduction of information and when intelligent mastery of a body of knowledge were considered. (26:352)

Katona's research revealed that the sequence AB was superior to BA when A was well organized and structurally clear. The material A was understood and its organization influenced the subsequent learning process. In the sequence AB a transfer effect occurred in which the learning process A determined the way in which material B was learned. Furthermore, it appeared that the principles acquired in A persisted in an adaptable form and were enriched and reinforced by the later B. For the sequence BA, the same conclusions did not hold. Katona remarked:

> We may conclude that the thoroughly intelligible material, if given first, provides the learner with an adaptable frame of reference, a sort of blueprint, containing the main structural features of the subject-matter. Such a framework can easily be complemented with individual data, which will be understood as fitting into the frame as parts determined by the whole. The frame itself can be strengthened and amplified by the inclusion of further parts. (26: 352)

Katona concluded that the reverse order produced a less meaningful enumeration of facts without structure and adding the later meaningful context was less efficient because the task of organization of the material was much more difficult. Thus, learning by understanding facilitates the subsequent learning of related knowledge by promoting organization of the later material in an already existing framework.

Transfer of learning. – An important aspect of the curriculum is the extent to which it provides the pupil with learning which will be applicable to wide areas of experience. A curriculum which provides for transfer must not emphasize specific experiences. Instead, it must provide organized experiences from which relationships, generalizations and concepts will be understood and be recognized for their implications for new situations.

Of course facts can not be dispensed with. Buswell makes an important comment in this regard:

> Enough facts or concretes must be presented to make the generalization or theory or relationship clear. Additional specifics are superfluous; too few are likely to result in verbalisms, memorized but not understood. There are certain other aspects of the technique of transfer which are well known. The generalization (law, rule, principle, relationship) to be transferred must appear as a constant factor in a number of variable situations; it must be pointed out or emphasized in some manner so that the learner is aware of it; and there must be some practice in applying the generalization to new situations. (6:461)

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Los Angeles City College

DIFFERENTIAL EQUATIONS EXHIBITING DIMENSIONAL HOMOGENEITY

M. S. Krick

Elementary textbooks on differential equations usually discuss methods for obtaining primitives of first order homogeneous equations. Another class of equations which admit of an elementary treatment and which are closely related to the homogeneous type will now be briefly discussed.

A function $\phi(x, y)$ is said to be homogeneous of dimension p if, for the parameter λ and some constant n,

$$\phi(\lambda x, \lambda^n y) = \lambda^p \phi(x, y) ,$$

where the constant n is termed the dimension of y; the dimension of x is unity. The first order differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be homogeneous of dimension p if

$$[M(\lambda x, \lambda^n y)](\lambda dx) + [N(\lambda x, \lambda^n y)](\lambda^n dy) = \lambda^p [M(x, y)dx + N(x, y)dy] .$$

Setting $\lambda = x^{-1}$ in this equation produces

$$x^{p-1}M(1, \frac{y}{x^n})dx + x^{p-n}N(1, \frac{y}{x^n})dy = M(x, y)dx + N(x, y)dy ,$$

so that the given equation can be written

$$x^{p-1}M(1, \frac{y}{x^n})dx + x^{p-n}N(1, \frac{y}{x^n})dy = 0 .$$

The substitution $y = vx^n$ then yields an equation with variables separable, which has the primitive

$$\text{(I)} \qquad \int \frac{dx}{x} + \int \frac{N(1, v)dv}{M(1,v) + nvN(1, v)} = \text{const.}$$

The original variables can be recovered by substituting $v = yx^{-n}$ in (I) after the integrations have been performed.

Next consider the function $\phi = \phi(u, v)$, where $u = \lambda x$ and $v = \lambda^n y$, and where $\phi(x, y)$ is homogeneous of dimension p when x and y have the dimensions l and n, respectively. Then

$$\frac{\partial\phi}{\partial\lambda} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial\lambda} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial\lambda} = x\frac{\partial\phi}{\partial u} + n\lambda^{n-1}y\frac{\partial\phi}{\partial v}$$

and also

$$\frac{\partial\phi}{\partial\lambda} = \frac{\partial}{\partial\lambda}[\phi(\lambda x, \lambda^n y)] = \frac{\partial}{\partial\lambda}[\lambda^p\phi(x, y)] = p\lambda^{p-1}\phi(x, y) .$$

Hence when $\lambda = 1$, it follows that

$$\text{(II)} \qquad x\frac{\partial\phi}{\partial x} + ny\frac{\partial\phi}{\partial y} = p\phi .$$

It is now easily shown that an integrating factor for the equation

$$M(x, y)dx + N(x, y)dy = 0 ,$$

which is homogeneous of dimension p when x and y have the dimensions l and n, respectively, is

$$\text{(III)} \qquad \mu = \frac{1}{Mx + nNy} ,$$

provided that $Mx + nNy \neq 0$, identically. The condition of integrability

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+nNy}\right) = \frac{\partial}{\partial x}\left(\frac{N}{Mx+nNy}\right)$$

is seen to hold after carrying thru the differentiations and applying **(II)** to the result, observing that $M(x, y)$ is homogeneous of dimension $(p-1)$ and that $N(x, y)$ is homogeneous of dimension $(p-n)$.

If the equation

$$Mdx + Ndy = 0$$

satisfies the homogeneity restrictions of the last paragraph and is also exact, then its primitive is

$$\text{(IV)} \qquad Mx + nNy = \text{const.} ;$$

for if

$$\Phi(x, y) = Mx + nNy = \text{const.} ,$$

then evaluating $d\Phi$ using the condition of integrability

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and the relation (II), observing that M and N are homogeneous of dimensions $(p-1)$ and $(p-n)$, respectively produces

$$d\Phi = p(Mdx + Ndy) = 0 ,$$

so that if $p \neq 0$,

$$Mdx + Ndy = 0 .$$

The equation

$$yF(xy)dx + xG(xy)dy = 0 ,$$

where F and G are functions of xy, is an example of the dimensionally homogeneous equation if x and y have the dimensions of 1 and -1, respectively. It is therefore reduced to an equation with separable variables by the substitution $y = vx^{-1}$, or it has the integrating factor

$$\mu = \frac{1}{xy(F-G)} ,$$

if $F \neq G$, identically.

In this discussion x has been assigned the dimension l, and y the dimension n. For purposes of integration, however, it is sometimes more convenient to reverse the choice of dimensions. Then equations (I) thru (IV) become

(I)′ $$\int \frac{dy}{y} + \int \frac{M(v,1)dv}{N(v,1) + nvM(v,1)} = \text{const.} , \quad v = xy^{-n}$$

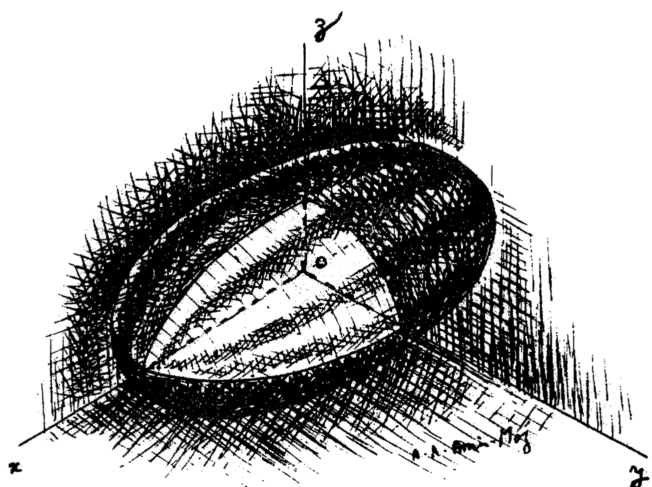
(II)′ $$nx\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} = p\phi$$

(III)′ $$\mu = \frac{1}{nMx + Ny}$$

(IV)′ $$nMx + Ny = \text{const.}$$

Finally, setting $n = 1$ in all of the above equations produces results which are applicable to equations which are homogeneous in the ordinary sense; in particular, (II) reduces to Euler's theorem of homogeneous functions.

Albright College,
Penna.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(See back cover for a representation of a hyperboloid of one sheet and a hyperboloid of two sheets.)

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

CURVILINEAR PROJECTION

Ali R. Amir-Moéz

INTRODUCTION : Let p be a pencil of straight lines. It is a well-known fact that if any line $d \not\in p$ intersects four fixed elements of p, the cross ratio of the four points of intersection is independent of the position of d. In this note we describe a family of curves having a similar property.

I. CROSS RATIO THROUGH A POINT O : Let $A, B, C,$ and D be four points of a plane. Then we define the cross ratio of $A, B, C, D,$ through O, a point of the plane ABC, to be that of the four straight lines $OA, OB, OC, OD,$ and we denote it by $O(A\,B\,C\,D)$. Note that A, B, C, D need not be collinear.

II. THEOREM : Let the equation of a two parameter family of plane curves be

$$(1) \qquad \gamma \equiv F(x, y) + \alpha x + \beta y = 0, \quad F(0, 0) = 0 .$$

Since the parameters α and β appear linearly in (1), given two points not collinear with the origin, one and only one curve of the family (1) is determined. Fix one point for (1) so that it depends on one and only one parameter. The equation of this one parameter family is

$$(2) \qquad \gamma(m) \equiv F(x, y) + mx + \left[\frac{-F(c, d)}{d} - \frac{c}{d}m\right] y = 0 ,$$

where (c, d) is the fixed point. Then

(a) The points of intersection of any two curves of the family (1) are collinear with the origin. (This part has been suggested by Professor P. H. Daus, UCLA)

(b) Given $m_1, m_2, m_3,$ and m_4, let $\{M\}_1, \{M\}_2, \{M\}_3, \{M\}_4$ be respectively the sets of the points of intersection of γ and $\gamma(m_1), \gamma(m_2), \gamma(m_3),$ and $\gamma(m_4)$. Let $M_1 \in \{M\}_1, M_2 \in \{M\}_2, M_3 \in \{M\}_3,$ and $M_4 \in \{M\}_4$. Then $O(M_1 M_2 M_3 M_4)$ is independent of the position of γ where O is the origin. Note that imaginary points are also included.

PROOF : To get the points of intersection of two curves of the family (1), we have to solve

$$(3) \qquad \begin{cases} F(x, y) + \alpha_1 x + \beta_1 y = 0 \\ F(x, y) + \alpha_2 x + \beta_2 y = 0 . \end{cases}$$

Subtracting these equations we get

$$(\alpha_1 - \alpha_2)x + (\beta_1 - \beta_2)y = 0 ,$$

which proves (a).

Now to prove (b) we observe that

$$(4) \qquad \begin{cases} F(x, y) + \alpha x + \beta y = 0 \\ F(x, y) + mx + \left[-\dfrac{F(c, d)}{d} - \dfrac{c}{d}m \right] y = 0 \end{cases}$$

can be replaced by

$$(5) \qquad \begin{cases} F(x, y) + \alpha x + \beta y = 0 \\ (\alpha - m)x + \left[\beta + \dfrac{F(c, d)}{d} + \dfrac{c}{d}m \right] y = 0 . \end{cases}$$

This suggests that $O(M_1 M_2 M_3 M_4)$ is the same as the cross ratio of the following straight lines

$$(\alpha - m_1)x + \left[\beta + \frac{F(c, d)}{d} + \frac{c}{d}m_1 \right] y = 0 ,$$

$$(\alpha - m_2)x + \left[\beta + \frac{F(c, d)}{d} + \frac{c}{d}m_2 \right] y = 0 ,$$

$$(\alpha - m_3)x + \left[\beta + \frac{F(c, d)}{d} + \frac{c}{d}m_3 \right] y = 0 ,$$

and

$$(\alpha - m_4)x + \left[\beta + \frac{F(c, d)}{d} + \frac{c}{d}m_4 \right] y = 0 ,$$

which is $(m_1 m_2 m_3 m_4)$.

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Queens College
Flushing 67, New York

THE OSCULATING PARABOLA TO ANY CURVE – $y = f(x)$

E. F. Canaday

Pass a parabola through the points P_1, P_2, P_3 and P_4 of the curve $y = f(x)$. Represent the directrix by the equation $y = mx + b$, and the focus by point (h, k).

The equation of the parabola is then

$$\frac{mx - y + b}{\sqrt{1+m^2}} = \sqrt{(x-h)^2 + (y-k)^2}$$

Since $y = f(x)$, we denote

$$(1+m)^2[(x-h)^2 + (y-k)^2] - (mx-y+b)^2$$

by $F(x)$. Now $F(x) = 0$ for P_1, P_2, P_3 and P_4, and hence (by Rolle's Theorem) $F'(x) = 0$ for P_5, P_6, and P_7, lying between P_1, P_2, P_3 and P_4 respectively. Likewise $F''(x) = 0$ for P_8 and P_9 lying between P_5, P_6 and P_7. Also $F'''(x) = 0$ for P_{10} between P_8 and P_9. Let all these points approach a point $P(x, y)$ and then m, b, h and k must all satisfy the equations $F(x) = 0$, $F'(x) = 0$, $F''(x) = 0$ and $F'''(x) = 0$. These are as follows:

(1) $\quad (1+m^2)[(x-h)^2+(y-k)^2] - m^2x^2 - y^2 - b^2 + 2mxy - 2mbx + 2by = 0$

(2) $\quad (1+m^2)[(x-h)+(y-k)y'] - m^2x - yy' + mxy' + my - mb + by' = 0$

(3) $\quad (1+m^2)[1+(y-k)y''+y'^2] - m^2 - yy'' - y'^2 + mxy'' + 2my' + by'' = 0$

(4) $\quad (1+m^2)[(y-k)y''' + 3y'y''] - yy''' - 3y'y'' + mxy''' + 3my'' + by''' = 0$

From equations (3), (4) and (2) respectively we get:

(5) $$y - k = \frac{1}{1+m^2}\left(y - mx - b - \frac{2my' + m^2y'^2 + 1}{y''}\right)$$

(6) $$y - k = \frac{1}{1+m^2}\left(y - mx - b - \frac{3my''(1+my')}{y'''}\right)$$

(7) $$y - k = \frac{1}{1+m^2}\left(y - mx - b - \frac{my + x - mb - h - hm^2}{y'}\right),$$

now equate right hand members of (5) and (6) and

(8) $$m^2(3y'y''^2 - y'^2y''') + m(3y''^2 - 2y'y''') - y''' = C.$$

The solution is

(9) $$m = \frac{y'''}{3y''^2 - y'y'''} \quad \text{(also an extraneous root } m = \frac{-1}{y'}\text{)}$$

Now substitute the value of $y-k$ from (6) into (2) and solve for $(x-h)$. Next put the values of $(y-k)$, $(x-h)$ and m into equation (1) and solve for b.

$$b = y - \left(\frac{2xy''' - 3y''(y'^2+1)}{2(3y''^2 - y'y''')}\right) \tag{10}$$

We now solve for h and k. Use the values found for m and b to form an expression for $y-mx-b$. Insert this value in (6) and solve for k and we get:

$$k = y - \left(\frac{3my''(y'^2 - 2my' - 1)}{2(1+m^2)y'''}\right). \tag{11}$$

Insert the value of m from (9) and

$$k = y - \left(\frac{3y''(3y'^2 y''^2 - y'^3 y''' - 3y''^2 - y'y''')}{2([3y''^2 - y'y''']^2 + y'''^2)}\right) \tag{12}$$

To find h substitute the value of $(y-k)$ from (6) into (2) and solve for $(x-h)$ and

$$x - h = \left(\frac{1}{1+m^2}\right)\left(-m(y-mx-b) + \frac{3my'y''(1+my')}{y'''}\right) \tag{13}$$

Now put the values of $(y-k)$ from (6) and $(x-h)$ from (13) into (1) and solve for $(y-mx-b)$ and we get

$$y - mx - b = \frac{3my''(y'^2+1)}{2y'''} \tag{14}$$

Now put this value of $(y-mx-b)$ and the value of m from (9) into (13) and solve for h.

We than have the following values of h, k, m and b and can write the desired equation of the osculating parabola. These values are as follows:

$$h = x - \frac{3}{2}\left(\frac{-y''y'''(1+y'^2) + 6y'y''^3}{(3y''^2 - y'y''')^2 + y'''^2}\right) \tag{15}$$

$$k = y - \left(\frac{3y''(3y'^2 y''^2 - y'^3 y''' - 3y''^2 - y'y''')}{2[(3y''^2 - y'y''')^2 + y'''^2]}\right) \tag{12}$$

$$m = \frac{y'''}{3y''^2 - y'y''} \tag{9}$$

$$b = y - \frac{2xy''' + 3y''(y'^2+1)}{2(3y''^2 - y'y''')} \tag{10}$$

Illustration (1). Applying the above formulas to find the osculating parabola to $y = x^3$ at the point (1, 1) we get the directrix $x - 15y - 1 = 0$ and the focus, (79/226, 55/226).

Illustration (2). Find the osculating parabola to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, at x_1, y_1.

(17) The directrix is $2(x_1x+y_1y) = x_1^2+y_1^2+a^2+b^2$.

(18) The focus is $h = \dfrac{x_1}{2}\left(1+\dfrac{a^2-b^2}{x_1^2+y_1^2}\right)$; $k = \dfrac{y_1}{2}\left(1+\dfrac{b^2-a^2}{x_1^2+y_1^2}\right)$.

(19) Equation of parabola: $(y_1x-x_1y)^2+2(b^2x_1x+a^2y_1y)-2a^2b^2 = 0$.

(20) The axis of the parabola is $y_1x-x_1y = -\dfrac{(b^2-a^2)x_1y_1}{x_1^2+y_1^2}$.

It is interesting to note that the axis of the osculating parabola to the ellipse at any point is parallel to the radius vector from the center to that point.

(21) Distance from vertex to focus is $\dfrac{a^2b^2}{2(x_1^2+y_1^2)^{3/2}}$:

coordinates of vertex are

(22) $$x = \frac{x_1}{2}\,\frac{(x_1^2+y_1^2)(x_1^2+y_1^2+a^2)+y_1^2(a^2-b^2)}{(x_1^2+y_1^2)^2}$$

(23) $$y = \frac{y_1}{2}\,\frac{(x_1^2+y_1^2)(x_1^2+y_1^2+b^2)+x_1^2(b^2-a^2)}{(x_1^2+y_1^2)^2}$$

To find the *locus of the focus* of the osculating parabola to the ellipse solve equations (18) for x_1 and y_1 and substitute these values into the equation of the ellipse for the desired relation between h and k, the equation of the curve.

Elimination of x and y in turn from equations (18) gives

(24) $$(y-k)^4-(y-k)^2(h^2-a^2+b^2-k^2)-h^2k^2 = 0$$

(25) $$(x-h)^4-(x-h)^2(k^2-b^2+a^2-h^2)-h^2k^2 = 0 .$$

These quadratic form equations may be solved for x and y and the results substituted into $b^2x^2+a^2y^2 = a^2b^2$ for the desired relation, the *locus of the focus*.

If we set $a = b = r$ the ellipse becomes a circle $x^2+y^2 = r^2$ and we find the directrix of the osculating parabola is at a distance $3r/2$ from the center of the circle, and the locus of the focus is $x^2+y^2 = r^2/4$. Now the vertex of the parabola coincides with the point (x_1, y_1) on the circle, a fact which is not true for the ellipse.

Meredith College
Raleigh, N. C.

ON INFINITE SUMS OF BESSEL FUNCTIONS

Leo Levi

The following equations, giving the sum of all orders of Bessel functions, are well known.

$$(A) \qquad \cos(a\sin x) = \sum_{n=-\infty}^{\infty} J_{2n}(a)\cos(2nx)$$

$$(B) \qquad \sin(a\sin x) = 2\sum_{n=0}^{\infty} J_{2n+1}(a)\sin[(2n+1)x]$$

Certain interesting relations may be derived as special cases of these expressions. These seem to be less known and with the exception of *(C)* have not been found in the literature by the writer.

Let p, q, r, s take on all values satisfying the relations $p = 4n$, $q = 4n+1$, $r = 4n+2$, $s = 4n+3$, resp.

Substituting $x = \pi$ into *(A)*, one obtains

$$(C) \qquad 1 = \sum_{n=-\infty}^{\infty} J_p(a) + \sum_{n=-\infty}^{\infty} J_r(a)$$

Substituting $x = \pi/2$ into *(A)*

$$(D) \qquad \cos a = \sum_{n=-\infty}^{\infty} J_p(a) - \sum_{n=-\infty}^{\infty} J_r(a)$$

Substituting $x = \pi/2$ into *(B)*

$$(E) \qquad \sin a = 2\sum_{n=0}^{\infty} J_q(a) - 2\sum_{n=0}^{\infty} J_s(a)$$

Adding *(C)* and *(D)*

$$(F) \qquad \tfrac{1}{2}(1+\cos a) = \sum_{n=-\infty}^{\infty} J_p(a)$$

Subtracting *(D)* from *(C)*

$$(G) \qquad \tfrac{1}{2}(1-\cos a) = \sum_{n=-\infty}^{\infty} J_r(a)$$

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80 Bennett Ave.
New York 33, N. Y.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

390. *Proposed by C. W. Trigg, Los Angeles City College.*

A regular tetrahedron and a regular octahedron have equal edges. Find the ratio of their volumes without computing the volume of either.

391. *Proposed by Melvin Hochster, Stuyvesant High School, New York.*

If

$$S_k = \sum_{i=0}^{2^{k-1}} \binom{2^k}{2i} n^{(2^{k-1}-i)}$$

and

$$T_k = \sum_{i=1}^{2^{k-1}} \binom{2^k}{2i-1} n^{(2^{k-1}-i)}$$

prove that the limit of the sequence S_k/T_k as $k \to \infty$ is $\sqrt{n}$ when $n > 0$.

392. *Proposed by Chih-yi Wang, University of Minnesota.*

A student used the incorrect formula $\text{Log}_e(m+n) = (\log_e m)(\log_e n)$ instead of the $\log_e(mn) = (\log_e m) + (\log_e n)$.

a). Show that the equation he used has at least one solution in n and m.

b). Are there general solutions of the equation $\log_e(x+y) = (\log_e x)(\log_e y)$?

393. *Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

Find a power series expansion of

$$P = \prod_{r=1}^{\infty} (1+x^{2^r}) \quad \text{for } |x| < 1 \text{ .}$$

394. *Proposed by Joseph Andrushkiw, Seton Hall University, New Jersey.*

Striking out a number of terms in the harmonic series without changing its order leaves a subseries (finite or infinite). Show that if r is any positive real number, there exist infinitely many subseries convergent to r.

395. *Proposed by Sidney Kravitz, Dover, New Jersey.*

It is well known that $f(n) = n^2 - n + 41$ yields prime numbers for $n \leq 40$. Show that $f(n)$ contains at most two prime factors for $n \leq 420$.

396. *Proposed by J. B. Love, Eastern Baptist College, Pennsylvania.*

Show that

$$\sum_{j=0}^{n-1} [x + j/n] = [nx]$$

where the brackets denote the greatest integer function.

SOLUTIONS

The Explorer

369. [March 1959] *Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

An explorer travels on the surface of the earth, assumed to be a perfect sphere, in the manner to be described. First, he travels 100 miles due north. He then travels 100 miles due east. Next he travels 100 miles due south. Finally, he travels 100 miles due west, ending at the point from which he started. Determine all the possible points from which he could have started.

Solution by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts. The problem here is to choose the starting point so that the two East-West legs of the tour, even though they differ by 100 miles of latitude, span the same number of longitudinal units. This can easily be done if the starting point is anywhere on the circle of South latitude which is exactly 50 miles below the equator.

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; J. M. Howell, Los Angeles City College, and the proposer.

A Circle of Appolonius

371. [March 1959] *Proposed by Leon Bankoff, Los Angeles, California.*

1) Find the radius of the circle which is tangent to two internally tangent circles and to their line of centers.
2) Construct the required circle, confining all construction lines within the larger circle.

I. *Solution by J. W. Clawson, Collegeville, Pennsylvania.* Let the circles be Γ_1, Γ_2, their centers C_1, C_2, their radii R, r ($R > r$). Also their diameters through O, their point of contact are OA and OB.

Invert with center O, radius $\sqrt{OA \cdot OB} = \sqrt{4Rr}$. Then, invert respectively into the tangent at B to Γ_2 and the tangent at A to Γ_1, while the line of centers inverts into itself.

Bisect AB at M, erect MN perpendicular to BA and lay off MN equal to BM. Then the circle, center N, radius MN, touches the three straight lines into which the circles and their line of centers invert. Lay off OD on the line of centers so that $OD \cdot OM = OA \cdot OB$. Join ON. Draw DC perpendicular to OA to meet ON at C. Then the circle with center C and **radius** CD is the circle required. (2)

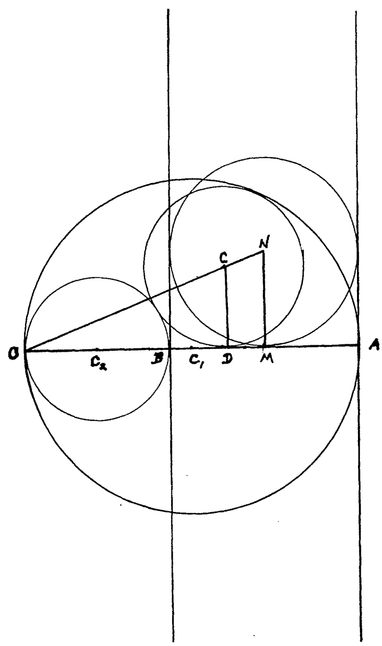
Now $OM = R + r$. Hence $OD = 4Rr/(R+r)$.

Also $MN = OA - OM = 2R - (R+r) = R - r$.

But $DC/MN = OD/OM = 4Rr/(R+r)^2$.

Hence the radius of the required circle is

$$\frac{4Rr(R-r)}{(R+r)^2}. \qquad (1)$$



II. *Solution by the proposer.* Let $r = AB/2 = AO$, $r_1 = AC/2 = AO_1$, where $r > r_1$, denote the radii of the given circles tangent internally at A. Let the required circle, with center ω and radius ρ, touch AB at H.

Apply the Heronian Formula, $S = \sqrt{s(s-a)(s-b)(s-c)}$, in the triangle $OO_1\omega$, in which $s = r$, $s - a = r_1$, $s - b = \rho$, $s - c = r - r_1 - \rho$.

Since $s = \rho(r - r_1)/2$, we have

$$\rho(r-r_1)/2 = \sqrt{rr_1\rho(r-r_1-\rho)}$$

or

$$\rho(r-r_1)^2 = 4rr_1(r-r_1-\rho)$$

which yields

$$\rho = 4rr_1(r-r_1)/(r+r_1)^2$$

2) We can devise a variety of constructions from the foregoing solutions. Note that $AH = 4rr_1(r+r_1)$ = twice the Harmonic Mean of AO and AO_1; $CH = 2r_1(r-r_1)/(r+r_1)$ = half the Harmonic Mean of r_1 and $(r-r_1)$; also that ω lies on the line joining A to the midpoint M of the semi-circumference above diameter CB. We can also make use of the observation that $CE = CH$, where E is the intersection of AM with CD.

An example of a construction confined within the circle (O) is as follows:

a) Construct the perpendicular bisector of CB and let it meet the semi-circumference on CB in the point M.

b) Erect a perpendicular to AB at C, cutting AM at E.

c) Lay off $CH = CE$ on CB.

d) Erect a perpendicular to CB at H, cutting AM at ω, the center of the required circle, whose radius $\rho = \omega H$.

Another way of locating H is to erect a perpendicular to AB at C, meeting the outer circle at D. Construct angle ADH = angle AO_2D, where O_2 is the midpoint of CB. This procedure stems from the fact that, in similar triangles AHD, AO_2D we have $AH/AD = AD/AO_2$ or $AH = AD^2/AO_2 = AC\cdot AB/AD_2 = 4rr_1(r+r_1)$.

Also solved by Norman Anning, Alhambra, California; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; Christopher Henrich, East Aurora High School, New York, and the proposer (two additional solutions).

A Trigonometric Identity

372. [March 1959] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Prove the identity

$$\sin^2(\theta_1+\theta_2+\cdots+\theta_n) = \sin^2\theta_1+\cdots+\sin^2\theta_n + 2\sum_{1\leq i<j\leq n}^{n} \sin\theta_i\sin\theta_j\cos(\theta_1+2\theta_{i+1}+\cdots+2\theta_{j-1}+\theta_j).$$

Solution by the proposer. We proceed by induction. The equality holds for $n = 1$ and $n = 2$. Let the property be true for $n = p$. Then setting

$$\theta = \theta_1+\cdots+\theta_p$$

it will suffice to prove the equality obtained by subtraction

$$\sin^2(\theta+\theta_{p-1}) - \sin^2\theta = \sin^2\theta_{p-1} - 2\sum_{i=1}^{p}\sin\theta_1\sin\theta_{p+1}\cdot\cos(\theta_i+2\theta_{i+1}+\cdots+2\theta_p+\theta_{p+1}).$$

The left hand side, A, is seen to be equal to

$$A = \sin\theta_{p+1}\sin(2\theta+\theta_{p+1})$$

The right hand side, B, is equal to

$$B = \sin^2\theta_{p+1} + \sin\theta_{p+1}\sum_{i=1}^{p}2\sin\theta_1\cos(\theta_i+2\theta_{i+1}+\cdots+2\theta_p+\theta_{p+1})$$

$$= \sin^2\theta_{p+1} + \sin\theta_{p+1}\sum_{i=1}^{p}[\sin(2\theta_i+2\theta_{i+1}+\cdots+2\theta_p+\theta_{p+1})$$

$$-\sin(2\theta_{i+1}+\cdots+2\theta_p+\theta_{p+1})]$$

$$= \sin^2\theta_{p+1} + \sin\theta_{p+1}[(\sin(2\theta_1+2\theta_2+\cdots+2\theta_p+\theta_{p+1}) - \sin\theta_{p+1}]$$

$$= \sin\theta_{p-1}\cdot\sin(2\theta_1+\cdots+2\theta_p+\theta_{p+1}) = A$$

The equality $A = B$ proves that the equality holds for $n = p+1$. The result follows by induction.

A Prime Number

373. [March 1959]. *Proposed by Edgar Karst, Endicott, New York.*

Prove or disprove the statement: "If $2100n+x$ is prime, then x is prime where x is a two digit number, n is a natural number, and 01 is considered as prime.

Solution by Harry M. Gehman, University of Buffalo. If x is a composite number between 1 and 100, then x is divisible by 2, 3, 5, or 7. Since each of these primes divides 2100, it follows that $2100n+x$ is composite. Hence if $2100n+x$ is prime, then x is prime. The same argument shows that if $210n+x$ is prime, then x is prime. In general, the above type of argument shows that if P_k is the product of the first k primes, and $1 < x < p_{k+1}^2$, and P_kn+x is prime, then x is prime.

Also solved by Leon Bankoff, Los Angeles, California; D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; Sidney Kravitz, Dover, New Jersey; Chih-yi Wang, University of Minnesota; and Dale Woods, Idaho State College.

Equivalent Triangles

374. [March 1959] *Proposed by Victor Thebault, Tennie, Sarthe, France.*

If an arbitrary straight line d, passing through any point P of the plane of a triangle ABC, meets the straight lines BC, CA and AB in points A_1,

B_1 and C_1, and the points obtained in prolonging the segments A_1P, B_1P, and C_1P by three times their length are A_1', B_1', and C_1', then the mid-points of AA_1', BB_1' and CC_1', A_2, B_2, and C_2, respectively, are the vertices of a triangle, the area of which is equal to that of triangle ABC.

I. *Solution by J. W. Clawson, Collegeville, Pennsylvania.* We take d for the X-axis, P for the origin of a system of rectangular Cartesian coordinates. Let A_1, B_1, C_1 be respectively $(a,0)$, $(b,0)$, $(c,0)$. Then, taking the slopes of AB, BC, CA to be respectively r, p, q, their equations are $y = r(x-c)$, $y = p(x-a)$, $y = q(x-b)$. Then points A, B, C are easily found to be $(\frac{bq-cr}{q-r}, \frac{(b-c)qr}{q-r})$ and two symmetrical expressions. Then A_1', B_1', C_1' are $(-3a, 0)$, $(-3b, 0)$, $(-3c, 0)$. Hence A_2 is $(\frac{(b-3a)q+(3a-c)r}{2(q-r)}, \frac{(b-c)qr}{2(q-r)})$, with symmetrical expressions for B_2 and C_2.

Using this notation, the area of each of the triangles ABC and $A_2B_2C_2$ becomes

$$\frac{[(a-b)pq+(b-c)qr+(c-a)pq]^2}{2(p-q)(q-r)(r-p)}.$$

II. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.* We may express the relations between the points by vectorial equalities and arrive at the desired result by vectorial multiplication. We first note that the point P is not necessarily on d. According to the notations as stated, we have

$$\vec{PA_1'} = -3\vec{PA_1} \quad \text{and} \quad 2\vec{PA_2} = \vec{PA} + \vec{PA_1'} = \vec{PA} - 3\vec{PA_1}$$

Now

$$2\vec{A_2B_2} = 2(\vec{PB_2} - \vec{PA_2}) = (\vec{PB} - 3\vec{PB_1} - \vec{PA} + 3\vec{PA_1})$$

$$2\vec{A_2B_2} = \vec{AB} - 3\vec{A_1B_1}$$

and similarly

$$2\vec{A_2C_2} = \vec{AC} - 3\vec{A_1C_1}\ .$$

Multiplying the last two equalities member to member and denoting by $\overline{ABC}$ the area of the oriented triangle ABC we get

$$4\overline{A_2B_2C_2} = (\vec{AB} - 3\vec{A_1B_1}) \times (\vec{AC} - 3\vec{A_1C_1})$$

$$= \vec{AB} \times \vec{AC} - 3(\vec{AB} \times \vec{A_1C_1} + \vec{A_1B_1} \times \vec{AC}) + 9\vec{A_1B_1} \times \vec{A_1C_1}$$

The last term being zero

$$4\overline{A_2B_2C_2} = \overline{ABC} - 3(\vec{AB} \times \vec{A_1A} + \vec{A_1A} \times \vec{AC})$$

$$= \overline{ABC} - 3(\vec{AB} - \vec{AC}) \times \vec{A_1A}$$

$$= \overline{ABC} + 3\vec{BC} \times \vec{CA} = 4\overline{ABC} \quad \text{Q.E.D.}$$

This problem may be generalized as follows: If $A_1B_1C_1$ is an inscribed triangle of ABC and if $\vec{PA_1'} = -n\vec{PA_1}$ (in the present case $n = 3$), $\vec{AA_2} = m\vec{AA_1'}$ (in the present case $m = ½$), then we have

$$\overline{A_2B_2C_2} = (1-m)(1-2m+mn)\overline{ABC} + m^2(1-m)^2\overline{A_1B_1C_1}$$

Also solved by Christopher Henrich (partially) and the proposer.

A Binomial Coefficient Relation

375. [March 1959] *Proposed by D. A. Steinberg, University of California Radiation Laboratory.*

Let m and n be positive integers.

1) If mn is odd then:

$$1 + \sum_{k=1}^{\frac{mn-1}{2}} \sum_{j=1}^{k} \binom{n}{j}\binom{k-1}{j-1} = \frac{(m+1)^n}{2}$$

2) If mn is even then:

$$1 + \sum_{k=1}^{\frac{mn-2}{2}} \sum_{j=1}^{k} \binom{n}{j}\binom{k-1}{j-1} + ½\sum_{j=1}^{\frac{mn}{2}} \binom{n}{j}\binom{\frac{mn-2}{2}}{j-1} = \frac{(m+1)^n}{2}$$

Comments by Henry W. Gould, West Virginia University. The problem as stated is incorrect. I shall give a counterexample and follow with a derivation of the correct expression for the summation in question.

The problem proposed by Mr. Steinberg says in part (1): Let m and n be positive integers. Then if mn is odd

$$1 + \sum_{k=1}^{\frac{mn-1}{2}} \sum_{j=1}^{k} \binom{n}{j}\binom{k-1}{j-1} = \frac{(m+1)^n}{2}$$

A counterexample to this is afforded by choosing $m = 5$ and $n = 3$, so the product $mn = 15$, an odd integer. The proposed value for the summation is then $\frac{(5+1)^3}{2} = \frac{216}{2} = 108$. However, for the summation I find the following calculations:

$$\frac{mn-1}{2} = \frac{14}{2} = 7. \quad \text{So} \quad S = \sum_{k=1}^{7} \sum_{j=1}^{k} \binom{3}{j}\binom{k-1}{j-1}$$

$$= \sum_{j=1}^{1}\binom{3}{j}\binom{0}{j-1} + \sum_{j=1}^{2}\binom{3}{j}\binom{1}{j-1} + \sum_{j=1}^{3}\binom{3}{j}\binom{2}{j-1} + \sum_{j=1}^{4}\binom{3}{j}\binom{3}{j-1} +$$

$$\sum_{j=1}^{5}\binom{3}{j}\binom{4}{j-1} + \sum_{j=1}^{6}\binom{3}{j}\binom{5}{j-1} + \sum_{j=1}^{7}\binom{3}{j}\binom{6}{j-1} .$$

From this we have,

$$s = 3 + [3+3] + [3+3(2)+1] + [3+3(3)+3] + [3+3(4)+6] + [3+3(5)+10] + [3+3(6)+15]$$

$$= 9(3) + 3[2+3+4+5+6] + 1 + 6 + 10 + 15 + 3[29] + 32 = 87 + 32 = 119$$

Adding 1 to this number I obtain 120 as the value, then, of the left-hand member of the relation in the problem, not 108. I note that these numbers are of the same order of magnitude and for various reasons I suspect that an incorrect formula followed from some miscalculation of combinatorial applications. It seems evident that the proposed relation is impossible just on the basis of the possibility of representing the right hand expression by a series of the type proposed. The best that can be done with the proposed series is to express it in terms of binomial coefficients, not powers of numbers.

I shall now show that the correct value of the sum in the first part of the problem is the binomial coefficient.

$$\binom{n+\frac{mn-1}{2}}{n}$$

Thus, in the case I calculated above, we find for the sum

$$\binom{3+\frac{15-1}{2}}{3} = \binom{3+7}{3} = \binom{10}{3} = 120 .$$

I shall prove the more general result below:

THEOREM:

$$\sum_{k=1}^{p}\sum_{j=1}^{k}\binom{n}{j}\binom{k-1}{j-1} = \binom{n+p}{n} - 1 ; \quad p \geqq 1,\ n \geqq 1 .$$

Proof:

$$\sum_{k=1}^{p}\sum_{j=1}^{k}\binom{n}{j}\binom{k-1}{j-1} = \sum_{k=1}^{p}\sum_{j=0}^{k-1}\binom{n}{j+1}\binom{k-1}{j} = \sum_{k=0}^{p-1}\sum_{j=0}^{k}\binom{n}{j+1}\binom{k}{j}$$

$$= \sum_{j=0}^{p-1}\binom{n}{j+1}\sum_{k=j}^{p-1}\binom{k}{j} = \sum_{j=0}^{p-1}\binom{n}{j+1}\binom{p}{j+1}$$

$$= \sum_{j=1}^{p} \binom{n}{j}\binom{p}{j} = \sum_{j=1}^{p} \binom{n}{j}\binom{p}{p-j} = \sum_{j=0}^{p} \binom{n}{j}\binom{p}{p-j} - 1$$

$$= \binom{n+p}{p} - 1 = \binom{n+p}{n} - 1 .$$

In this proof we needed only two easy lemmas, each well-known

$$\sum_{k=j}^{\propto} \binom{k}{j} = \binom{\propto+1}{j+1} \quad \text{and} \quad \sum_{j=0}^{p} \binom{x}{j}\binom{y}{p-j} = \binom{x+y}{p} .$$

This last summation is referred to in the mathematical literature as the Vandermonde convolution. Some papers of mine in the *American Mathematical Monthly* discuss a vast collection of generalizations of it.

To evaluate the proposed series, we let $p = \frac{mn-1}{2}$ in the theorem and so obtain the value

$$\binom{n+\frac{mn-1}{2}}{n}$$

which is clearly not in general equal to $\frac{(m+1)^n}{2}$.

Valuable references for formulas of the type we discuss are the following standard works:

Schwatt, Issac J., *INTRODUCTION TO THE OPERATIONS WITH SERIES,* University of Pennsylvania Press, 1924.

Hagen, John George, *SYNOPSIS DER HOHEREN MATHEMATIK*, Berlin, 1891.

Netto, Eugen, *LEHRBUCH DER COMBINATORIK*, Leipzig, 1901, republished by Chelsea Reprint Co., NY, 1957.

Part (2) of the proposer's problem would be handled in similar fashion.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 258. Each of the face angles of a pentahedral angle is 60°. Find the angle between two non-consecutive edges. [*Submitted by Melvin Hochster*].

Q 259. Show that if n is even, then

$$\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2}\cos(n-4)\theta + \cdots + \frac{{}_nC_{n/2}}{2}\right]$$

while if n is odd

$$\cos^n\theta = \frac{1}{2^{n-1}}[\cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2}\cos(n-4)\theta + \cdots + \frac{{}^nC_{(n-1)}}{2}\cos\theta].$$

[*Submitted by Ben B. Bowen*]

Q 260. A housewife purchased some sugar **for $2.16.** Had the sugar cost one cent a pound less, she would have received **3 pounds** more for the same expenditure. How many pounds did she buy? [*Submitted by C. W. Trigg*]

Q 261. A merchant who had x bags of flour, passed through thirteen gates of thirteen cities. The curious levy at each gate was one half of what he possessed at the moment. The officer in charge at each gate was his friend. After the levy was paid, each officer gave him one bag as a gift. To his surprise, he still had exactly x bags after he had completed his journey. Determine x. [*Submitted by Chih-yi Wang*]

Q 262. For what values of u_0 does the sequence $[u_n]$ diverge when $u_{n+1} = \frac{1}{u_n+2}$? [*Submitted by M. S. Klamkin*]

Comment on a Quicky

Q 32. [March 1951] Each edge of a cube is a one-ohm resistor. What is the resistance between two diagonally opposite vertices of the cube?

Alternate solution by C. W. Trigg. Consider planes passed through the extremities of each group of three resistors issuing from two opposite vertices. These planes divide the resistors into three series-connected sets of parallel resistors, consisting of 3, 6 and 3 resistors, respectively. Hence, the total resistance is

$$\frac{1}{3}+\frac{1}{6}+\frac{1}{3} \quad \text{or} \quad \frac{5}{6} \quad \text{or} \quad 0.833 \text{ ohms.}$$

Answers

A 258. Let A, B, C, D, and E be points, each on a differend edge so that $VA = VB = VC = VD = VE$ where V is the vertex. Triangles VAB and VAC are equilateral triangles, and since $VA = AB$, $VB = BC$, and $AC = AC$, we have triangle AVC congruent to triangle ABC. Thus, angle AVC equals angle ABC. But angle ABC is an angle of the regular pentagon $ABCDE$ and therefore equals 108°. Hence, angle AVC equals 108°.

A 259. Use the binomial expansion to express $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

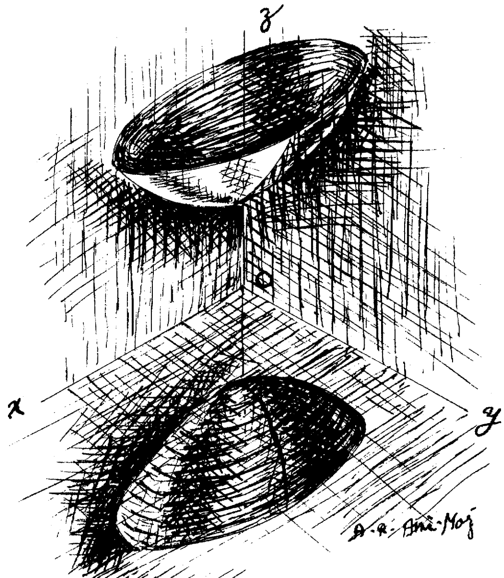
A 260. $216 = 2^3 \cdot 3^3 = 8 \cdot 27 = 9 \cdot 24$, so she bought 24 pounds.

A 261. He started with $x = 2$ bags.

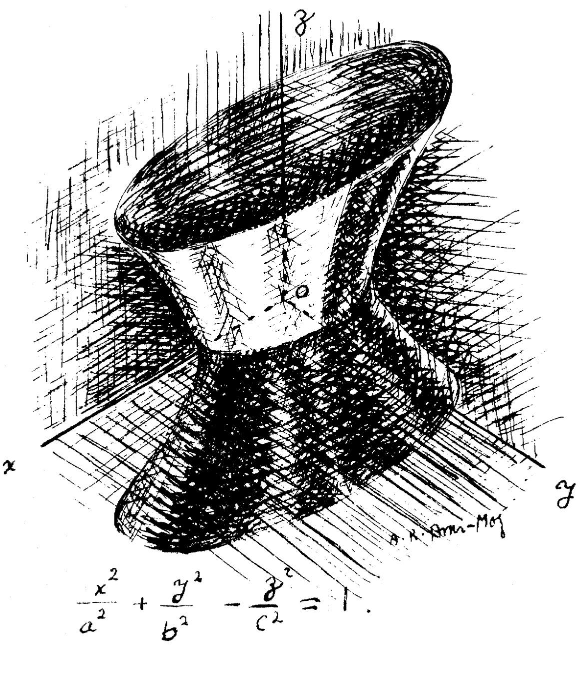
A 262. Consider the inverse sequence $a_n = \frac{1}{a_{n+1}+2}$ or $a_{n+1} = \frac{1}{a_n} - 2$ where $a_0 = -2$. Then $[u_n]$ diverges for $u_0 = a_r$, r arbitrary, since $u_r = -2$. $[u_n]$ can be shown to converge for all other real values of u_0.

Friends of the MATHEMATICS MAGAZINE will be interested in the fact that this magazine has made a test distribution through the newsstands beginning with last March-April issue and, up to the last report has sold 40% of the offerings, a gratifying surprize indeed.

—Ed.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1.$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

(See page 102 for a representation of an ellipsoid.)